

# Approximation of the high-frequency Helmholtz kernel by nested directional interpolation

Steffen Börm and Jens M. Melenk

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We present and analyze an approximation scheme for highly oscillatory kernel functions. This scheme is based on polynomial interpolation combined with suitable pre- and postmultiplication by plane waves. The scheme is shown to converge exponentially in the polynomial degree and supports multilevel approximation techniques. Our convergence analysis may be employed to establish exponential convergence of certain classes of fast methods for discretizations of the Helmholtz integral operator that feature polylogarithmic-linear complexity.

## 1. Introduction

Integral operators with highly oscillatory kernels arise, for example, in time-harmonic settings of acoustic and electromagnetic scattering. They have the form

$$u \mapsto \mathcal{G}[u](x) := \int_{\Gamma} k(x, y) u(y) dy, \quad (1.1)$$

where  $\Gamma$  is typically a subset of  $\mathbb{R}^n$  or a  $(n-1)$ -dimensional submanifold of  $\mathbb{R}^n$ . Prominent examples of kernels  $k$  that are studied in detail in the present work are the three- and two-dimensional Helmholtz kernels

$$g(x, y) = \frac{\exp(\mathbf{i}\kappa\|x - y\|)}{4\pi\|x - y\|}, \quad h(x, y) = \frac{\mathbf{i}H_0^{(1)}(\kappa\|x - y\|)}{4}, \quad (1.2)$$

where  $H_0^{(1)}$  is the first kind Hankel function of order 0. For real, large *wave numbers*  $\kappa \in \mathbb{R}_{\geq 0}$ , these two functions are highly oscillatory. A more general setting is one where the factorization

$$k(x, y) = \left[ k(x, y) \exp(-\mathbf{i}\kappa\|x - y\|) \right] \exp(\mathbf{i}\kappa\|x - y\|) \quad (1.3)$$

decomposes the kernel  $k$  into a highly oscillatory contribution  $\exp(\mathbf{i}\kappa\|x - y\|)$  and a non-oscillatory (or at least less oscillatory) term  $k(x, y)\exp(-\mathbf{i}\kappa\|x - y\|)$ . Again, the two- and three-dimensional Helmholtz kernels are precisely of this type.

Discretization of the integral operator (1.1) by Galerkin or collocation methods leads to fully populated system matrices. In order to reduce the complexity of these methods, various compression schemes have been devised in the past. Techniques that address the particular challenges arising in the high-frequency setting of large wave numbers  $\kappa$  include the *fast multipole method*, [22, 15, 12, 11], the *butterfly method*, [19, 10, 17], and *directional approximations*, [9, 14, 18, 2]. In many of these approaches, the underlying justification is based on approximating the kernel using suitable expansion systems; in the high frequency setting, these expansion systems are typically not polynomial in order to account for the oscillatory behavior of  $k$ . Polylogarithmic-linear complexity of the algorithms requires a second ingredient: a multilevel structure. Suitable expansion systems are given on each level and a fast transfer between levels has to be effected, e.g., by “re-expansion”.

From the point of view of numerical analysis, the stability of such an iterated approximation requires investigation. For a class of expansion systems that consist of products of plane waves and polynomials, we present a full analysis of the approximation properties and a stability analysis of the corresponding iterated re-expansion. We point out that underlying our approximations is polynomial interpolation (e.g., Chebyshev interpolation). This very powerful tool permits a full analysis, in particular of the algorithm of [18]. Many algorithms in the literature such as [2] enforce the required multilevel structure by algebraic means; while these algorithms can be very successful in practice, a complete analysis is still missing since similarly powerful analytical tools are not available. We defer a more detailed discussion of our problem setting to Section 2.2 after the necessary notation has been provided.

*Concerning notation:* We write  $\langle \cdot, \cdot \rangle$  for the *bilinear* form

$$\mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}, \quad (x, y) \mapsto \langle x, y \rangle := \sum_{j=1}^n x_j y_j. \quad (1.4)$$

This is *not* a sesquilinear form, but the restriction to the real subspace  $\mathbb{R}^n$  is the standard Euclidean inner product. Throughout,  $\|\cdot\|$  denotes the Euclidean norm on  $\mathbb{R}^n$ . For vectors  $c \in \mathbb{R}^n$  and functions  $u$ , we use the shorthand  $\exp(\mathbf{i}\kappa\langle c, \cdot \rangle)u$  to denote the function  $x \mapsto \exp(\mathbf{i}\kappa\langle c, x \rangle)u(x)$ . In the special case  $n = 1$ , we simply write  $\exp(\mathbf{i}\kappa c \cdot)u$  for the function  $x \mapsto \exp(\mathbf{i}\kappa c x)u(x)$ .

For compact sets  $B$ , we use the maximum norm

$$\|f\|_{\infty, B} := \max\{|f(x)| : x \in B\} \quad \text{for all } f \in C(B),$$

for bounded linear operators  $\Psi \in L(C(B_1), C(B_2))$  that map functions in  $C(B_1)$  to  $C(B_2)$ , we denote the induced operator norm by

$$\|\Psi\|_{\infty, B_2 \leftarrow B_1} := \max \left\{ \frac{\|\Psi[f]\|_{\infty, B_2}}{\|f\|_{\infty, B_1}} : f \in C(B_1) \setminus \{0\} \right\}.$$

## 2. Directional techniques for oscillatory functions

### 2.1. Polynomials and tensor interpolation

Since polynomial interpolation on tensor product domains features importantly in the present paper, let us fix some notation and assumptions. For  $m \in \mathbb{N}_0$ , we denote by  $\Pi_m$  the space of univariate polynomials of degree  $m$ . Let  $\xi_0, \dots, \xi_m \in [-1, 1]$  be distinct interpolation points and define the associated Lagrange polynomials by

$$L_\nu(z) := \prod_{\substack{\mu=0 \\ \mu \neq \nu}}^m \frac{z - \xi_\mu}{\xi_\nu - \xi_\mu} \quad \text{for all } \nu \in [0 : m], \ z \in \mathbb{C}. \quad (2.1)$$

The corresponding interpolation operator is given by

$$\mathfrak{I}: C[-1, 1] \rightarrow \Pi_m, \quad f \mapsto \sum_{\nu=0}^m f(\xi_\nu) L_\nu.$$

For a general interval  $[a, b] \subseteq \mathbb{R}$ , we introduce the affine mapping

$$\Phi_{[a,b]}: \mathbb{C} \rightarrow \mathbb{C}, \quad z \mapsto \frac{b+a}{2} + \frac{b-a}{2}z,$$

that takes  $[-1, 1]$  bijectively to  $[a, b]$ , and define the interpolation operator

$$\mathfrak{I}_{[a,b]}: C[a, b] \rightarrow \Pi_m, \quad f \mapsto \sum_{\nu=0}^m f(\xi_{[a,b],\nu}) L_{[a,b],\nu},$$

with the transformed interpolation points and Lagrange polynomials

$$\xi_{[a,b],\nu} := \Phi_{[a,b]}(\xi_\nu), \quad L_{[a,b],\nu} := L_\nu \circ \Phi_{[a,b]}^{-1} \quad \text{for all } \nu \in [0 : m]. \quad (2.2)$$

Tensor product interpolation on an axis-parallel  $n$ -dimensional box

$$B := [a_1, b_1] \times \dots \times [a_n, b_n]$$

is defined by combining transformed interpolation points and polynomials for the  $n$  coordinate intervals: we let

$$\begin{aligned} \xi_{B,\nu} &:= (\xi_{[a_1,b_1],\nu_1}, \dots, \xi_{[a_n,b_n],\nu_n}) \in B, \\ L_{B,\nu}(x) &:= L_{[a_1,b_1],\nu_1}(x_1) \cdots L_{[a_n,b_n],\nu_n}(x_n) \quad \text{for all } x \in \mathbb{C}^n, \ \nu \in M := [0 : m]^n \end{aligned}$$

and define the tensor interpolation operator by

$$\mathfrak{I}_B[f] = \mathfrak{I}_{[a_1,b_1]} \otimes \dots \otimes \mathfrak{I}_{[a_n,b_n]}[f] = \sum_{\nu \in M} f(\xi_{B,\nu}) L_{B,\nu} \quad \text{for all } f \in C(B). \quad (2.3)$$

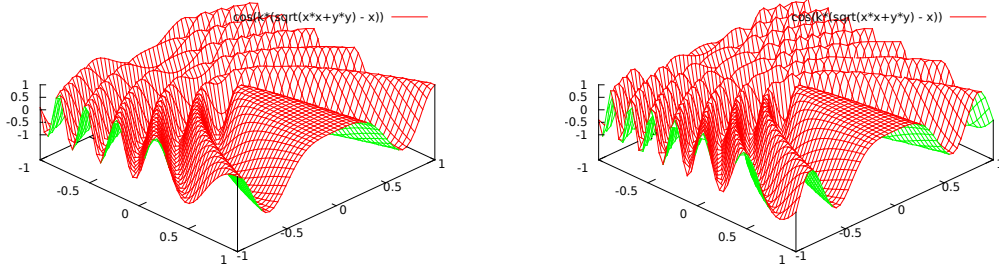


Figure 1:  $x \mapsto \cos(\kappa(\|x\| - x_1))$  in  $[-1, 1] \times [-1, 1]$  for  $\kappa \in \{15, 20\}$

## 2.2. Directional multilevel approximation of oscillatory functions

### 2.2.1. Directional single level approximation

The factorization (1.3) identifies the function

$$(x, y) \mapsto \exp(\mathbf{i}\kappa\|x - y\|)$$

as the oscillatory part of  $k$ . For large  $\kappa$ , a separable form cannot be obtained by standard polynomial approximation, as a large degree would be required. In order to overcome this obstacle, we follow an idea of Brandt [9], Engquist & Ying [14], Messner *et al.* [18] and construct a *directional* approximation: we introduce a vector  $c \in \mathbb{R}^n$  with  $\|c\| = 1$  and use  $\langle x - y, c \rangle$  as an approximation of  $\|x - y\|$ :

$$\begin{aligned} \exp(\mathbf{i}\kappa\|x - y\|) &= \exp(\mathbf{i}\kappa\langle x - y, c \rangle) \exp(\mathbf{i}\kappa(\|x - y\| - \langle x - y, c \rangle)) \\ &= \exp(\mathbf{i}\kappa\langle x, c \rangle) \overline{\exp(\mathbf{i}\kappa\langle y, c \rangle)} \exp(\mathbf{i}\kappa(\|x - y\| - \langle x - y, c \rangle)). \end{aligned} \quad (2.4)$$

Since the first factor depends only on  $x$  and the second only on  $y$ , a separable approximation of the third term in (2.4) gives rise to a separable approximation of the entire product. Figure 1 suggests that this term is a smooth function in a cone extending from zero in the chosen direction  $c$ , so that standard polynomial interpolation can be employed.

Specifically, given axis-parallel target and source boxes  $\tau, \sigma \subseteq \mathbb{R}^n$ , we approximate the directionally modified kernel function

$$k_c(x, y) := k(x, y) \exp(-\mathbf{i}\kappa\langle x - y, c \rangle) \quad (2.5)$$

by its interpolating polynomial

$$\tilde{k}_{c,\tau\sigma}(x, y) =: \sum_{\nu \in M} \sum_{\mu \in M} k_c(\xi_{\tau,\nu}, \xi_{\sigma,\mu}) L_{\tau,\nu}(x) L_{\sigma,\mu}(y) \quad \text{for all } x \in \tau, y \in \sigma. \quad (2.6)$$

Combining the approximation (2.6) of  $k_c$  with (2.4) leads to an approximation of the kernel function  $k$  by

$$\tilde{k}_{\tau\sigma}(x, y) = \sum_{\nu \in M} \sum_{\mu \in M} k_c(\xi_{\tau,\nu}, \xi_{\sigma,\mu}) L_{\tau c, \nu}(x) \overline{L_{\sigma c, \mu}(y)} \quad \text{for all } x \in \tau, y \in \sigma, \quad (2.7)$$

where the exponential factors are included in directionally modified Lagrange polynomials

$$\begin{aligned} L_{\tau c, \nu}(x) &:= \exp(\mathbf{i}\kappa\langle x, c \rangle) L_{\tau, \nu}(x), \\ L_{\sigma c, \mu}(y) &:= \exp(\mathbf{i}\kappa\langle y, c \rangle) L_{\sigma, \mu}(y) \end{aligned} \quad \text{for all } x \in \tau, y \in \sigma.$$

If  $M$  has only few elements, (2.7) is a short sum of *separated* functions. In fact, if the triple  $(\tau, \sigma, c)$  satisfies the *parabolic admissibility condition* (3.18), exponential convergence in the polynomial degree  $m$  can be expected. For the three- and two-dimensional Helmholtz kernels, this is proven in Corollaries 3.14 and 4.3. We mention in passing that this result is already sufficient to justify certain purely algebraic approximation schemes based on orthogonal factorizations [3] that depend only on the existence of degenerate approximation.

For an analysis of this single-level approximation, it is convenient to introduce the plane wave function

$$w_c: \tau \times \sigma \rightarrow \mathbb{C}, \quad (x, y) \mapsto \exp(\mathbf{i}\kappa\langle x - y, c \rangle) \quad (2.8)$$

since the approximation (2.7) of the kernel function  $k$  can be compactly written as

$$\tilde{k}_{\tau\sigma} = w_c \mathfrak{I}_{\tau \times \sigma} [\overline{w_c} k],$$

i.e., as the combination of multiplication operators and standard tensor interpolation.

Due to  $w_c(x, y) = \exp(\mathbf{i}\kappa\langle x - y, c \rangle) = \exp(\mathbf{i}\kappa\langle x, c \rangle) \exp(\mathbf{i}\kappa\langle y, -c \rangle)$ , we have

$$\tilde{k}_{\tau\sigma} = \mathfrak{I}_{\tau, c} \otimes \mathfrak{I}_{\sigma, -c} [k]$$

with

$$\begin{aligned} \mathfrak{I}_{\tau, c}[u] &:= \exp(\mathbf{i}\kappa\langle c, \cdot \rangle) \mathfrak{I}_{\tau} [\exp(\mathbf{i}\kappa\langle -c, \cdot \rangle) u] && \text{for all } u \in C(\tau), \\ \mathfrak{I}_{\sigma, -c}[u] &:= \exp(\mathbf{i}\kappa\langle -c, \cdot \rangle) \mathfrak{I}_{\sigma} [\exp(\mathbf{i}\kappa\langle c, \cdot \rangle) u] && \text{for all } u \in C(\sigma). \end{aligned}$$

### 2.2.2. Efficient multilevel matrix representation

The separable structure of the kernel approximation  $\tilde{k}_{\tau\sigma}$  can be exploited algorithmically. However, multilevel techniques have additionally to be brought to bear for the sake of efficiency. We describe here a variant that is used in [18]. To fix ideas, we consider a Galerkin discretization of the integral operator  $\mathcal{G}$  in the special case that  $\Gamma \subset \mathbb{R}^n$  is an  $(n-1)$ -dimensional manifold. The Galerkin method using a basis  $(\varphi_i)_{i \in \mathcal{I}}$  leads to the *stiffness matrix*  $G \in \mathbb{C}^{\mathcal{I} \times \mathcal{I}}$  given by

$$G_{ij} = \int_{\Gamma} \varphi_i(x) \int_{\Gamma} k(x, y) \varphi_j(y) dy dx \quad \text{for all } i, j \in \mathcal{I}. \quad (2.9)$$

The separable approximation (2.7) of the kernel  $k$  can only be used if  $k_c$  is sufficiently smooth in  $\tau \times \sigma$ , and this is the case if appropriate *admissibility conditions* hold (below, we will identify (3.18) as an appropriate one). In order to satisfy the admissibility condition, we recursively split the index set  $\mathcal{I}$  into disjoint subsets  $\hat{\tau}$  called *clusters* and construct axis-parallel *bounding boxes*  $\tau$  such that

$$\text{supp } \varphi_i \subseteq \tau \quad \text{for all } i \in \hat{\tau}.$$

We split  $\mathcal{I} \times \mathcal{I}$  into subsets  $\hat{\tau} \times \hat{\sigma}$  with clusters  $\hat{\tau}, \hat{\sigma}$  such that either

- $\hat{\tau}$  and  $\hat{\sigma}$  contain only a small number of indices, or
- the corresponding boxes  $\tau \times \sigma$  satisfy the admissibility condition (3.18).

In the first case, we store  $G|_{\hat{\tau} \times \hat{\sigma}}$  directly. In the second case, we use (2.7) to get

$$\begin{aligned} G_{ij} &= \int_{\Gamma} \varphi_i(x) \int_{\Gamma} k(x, y) \varphi_j(y) dy dx \approx \int_{\Gamma} \varphi_i(x) \int_{\Gamma} \tilde{k}_{\tau\sigma}(x, y) \varphi_j(y) dy dx \\ &= \sum_{\nu, \mu \in M} \underbrace{k_c(\xi_{\tau, \nu}, \xi_{\sigma, \mu})}_{=: S_{\tau\sigma, \nu\mu}} \underbrace{\int_{\Gamma} \varphi_i(x) L_{\tau c, \nu}(x) dx}_{=: v_{\tau c, i\nu}} \underbrace{\int_{\Gamma} \varphi_j(y) \overline{L_{\sigma c, \mu}(y)} dy}_{=: \overline{v_{\sigma c, j\mu}}} \\ &= (V_{\tau c} S_{\tau\sigma} V_{\sigma c}^*)_{ij} \quad \text{for all } i \in \hat{\tau}, j \in \hat{\sigma}. \end{aligned} \quad (2.10)$$

Since the *coupling matrices*  $S_{\tau\sigma} \in \mathbb{C}^{M \times M}$  are small, we can afford to store them explicitly.

The *basis matrices*  $V_{\tau c} \in \mathbb{C}^{\hat{\tau} \times M}$  are too large to be stored, but we can take advantage of the fact that the sets  $\hat{\tau}$  are nodes of a tree structure so as to obtain a hierarchical representation: if  $\hat{\tau}$  is a leaf of the tree, we assume that  $\hat{\tau}$  contains only a few indices and we can afford to store  $V_{\tau c}$  directly. If  $\hat{\tau}$  has a son  $\hat{\tau}'$ , we select a direction  $c'$  close to  $c$  and use interpolation to obtain

$$\begin{aligned} L_{\tau c, \nu}(x) &= \exp(\mathbf{i}\kappa\langle x, c \rangle) L_{\tau, \nu}(x) \\ &= \exp(\mathbf{i}\kappa\langle x, c' \rangle) \exp(\mathbf{i}\kappa\langle x, c - c' \rangle) L_{\tau, \nu}(x) \\ &\approx \exp(\mathbf{i}\kappa\langle x, c' \rangle) \sum_{\nu' \in M} \underbrace{\exp(\mathbf{i}\kappa\langle \xi_{\tau', \nu'}, c - c' \rangle) L_{\tau, \nu}(\xi_{\tau', \nu'})}_{=: e_{\tau' \tau c, \nu' \nu}} L_{\tau', \nu'}(x) \end{aligned} \quad (2.11)$$

and therefore

$$V_{\tau c}|_{\hat{\tau}' \times M} \approx V_{\tau' c'} E_{\tau' \tau c} \quad (2.12)$$

with the *transfer matrix*  $E_{\tau' \tau c} \in \mathbb{C}^{M \times M}$ . If we replace  $V_{\tau c}$  by an approximation  $\tilde{V}_{\tau c}$  given by  $\tilde{V}_{\tau c} = V_{\tau c}$  if  $\hat{\tau}$  is a leaf and

$$\tilde{V}_{\tau c}|_{\hat{\tau}' \times M} := \tilde{V}_{\tau' c'} E_{\tau' \tau c} \quad \text{for all sons } \hat{\tau}' \text{ of } \hat{\tau} \quad (2.13)$$

otherwise, we obtain the *directional  $\mathcal{H}^2$ -matrix approximation* of  $G$ .

**Remark 2.1 (Directions)** *In an algorithmic realization of directional  $\mathcal{H}^2$ -matrices, a finite set of directions is associated with each cluster  $\hat{\tau}$ . The necessary directional resolution is given by the admissibility condition (3.18b), and there are simple algorithms [3, 2] for constructing suitable directions.*

*In spite of the fact that large clusters require a large number of directions, it can be shown that in typical situations the resulting directional  $\mathcal{H}^2$ -matrices require  $\mathcal{O}(Nk + \kappa^2 k^2 \log N)$  units of storage, where  $N = \#\mathcal{I}$ ,  $k = \#M = (m+1)^n$  [3, 18]. If  $\mathcal{H}$ -matrices are used to treat the low-frequency case, we obtain a similar complexity estimate [2].*

**Remark 2.2 (Algebraic recompression)** *In the interest of efficiency, the algorithm in [18] combines the above techniques with further algebraic recompression of the coupling matrices.*

**Remark 2.3 (Nested multilevel)** *Nested multilevel structures are essential for poly-logarithmic-linear complexity in the high-frequency setting. Instead of resorting to interpolation to set up the factorizations (2.10) and (2.13), it is also possible to construct them directly via approximate rank-revealing factorizations based on a heuristic pivoting strategy as proposed in [2].*

### 2.2.3. Error analysis via multilevel approximation of the kernel

The goal of the article is to provide a rigorous error analysis for the various approximation steps in Section 2.2.2. In the present section, we set the stage for the error analysis by casting the analysis in the framework of polynomial interpolation.

Let  $(\tau, \sigma, c)$  satisfy the admissibility condition (3.18). We fix sequences

$$\tau_0 \supseteq \tau_1 \supseteq \cdots \supseteq \tau_L, \quad \sigma_0 \supseteq \sigma_1 \supseteq \cdots \supseteq \sigma_L$$

of axis-parallel boxes and a sequence  $c_0, \dots, c_L \in \mathbb{R}^n$  of directions such that

- $\tau_0 = \tau$ ,  $\sigma_0 = \sigma$ ,  $c_0 = c$ ,
- $\tau_\ell$  is a son of  $\tau_{\ell-1}$ ,  $\sigma_\ell$  is a son of  $\sigma_{\ell-1}$  for all  $\ell \in [1 : L]$ , and
- $\tau_L$  and  $\sigma_L$  are leaf clusters.

The re-interpolation (2.11) means that we replace

$$\begin{aligned} v_{\tau c, i\nu} &= \int_{\Gamma} \varphi_i(x) L_{\tau c, \nu}(x) dx && \text{by the approximation} \\ \tilde{v}_{\tau c, i\nu} &= \int_{\Gamma} \varphi_i(x) \underbrace{\mathfrak{I}_{\tau_L, c_L} \circ \cdots \circ \mathfrak{I}_{\tau_1, c_1}}_{=: \tilde{L}_{\tau c, \nu}(x)} [L_{\tau c, \nu}](x) dx && \text{for all } i \in \hat{\tau}_L, \nu \in M. \end{aligned} \quad (2.14)$$

Here we use the notation  $\tilde{L}_{\tau c, \nu}$  for the sake of brevity, in spite of the fact that it depends on the entire sequences  $\tau_0, \dots, \tau_L$  and  $c_0, \dots, c_L$ . In particular, different approximations

are used for different leaf clusters  $\tau_L$ . We can analyze the re-interpolation error by gauging the difference between  $L_{\tau c, \nu}$  and  $\tilde{L}_{\tau c, \nu}$ .

An alternative, very closely related approach to estimating the accuracy of the algorithm described in Section 2.2.2 is to study the effect of interpolating the kernel functions  $k$ : we denote the corresponding interpolation operators by

$$\mathfrak{I}_{\tau_\ell \times \sigma_\ell, c_\ell} := \mathfrak{I}_{\tau_\ell, c_\ell} \otimes \mathfrak{I}_{\sigma_\ell, -c_\ell} \quad \text{for all } \ell \in [0 : L], \quad (2.15)$$

and note that our nested interpolation scheme approximates  $k|_{\tau_L \times \sigma_L}$  by

$$\hat{k}_{\tau\sigma} := \mathfrak{I}_{\tau_L \times \sigma_L, c_L} \circ \cdots \circ \mathfrak{I}_{\tau_0 \times \sigma_0, c_0}[k]. \quad (2.16)$$

The analysis of the algorithm described in Section 2.2.2 amounts to estimating the error  $k - \hat{k}_{\tau\sigma}$ . Writing the error as a telescoping sum

$$k - \hat{k}_{\tau\sigma} = \sum_{\ell=0}^{L-1} \mathfrak{I}_{\tau_L \times \sigma_L, c_L} \circ \cdots \circ \mathfrak{I}_{\tau_{\ell+1} \times \sigma_{\ell+1}}[k - \mathfrak{I}_{\tau_\ell \times \sigma_\ell, c_\ell}[k]]$$

splits the error analysis into two parts: the analysis of the interpolation errors  $k - \mathfrak{I}_{\tau_\ell \times \sigma_\ell, c_\ell}[k]$  that is the topic of Sections 3 and 4, and a stability analysis of the nested interpolation operators  $\mathfrak{I}_{\tau_L \times \sigma_L, c_L} \circ \cdots \circ \mathfrak{I}_{\tau_{\ell+1} \times \sigma_{\ell+1}, c_{\ell+1}}$  for all  $\ell \in [0 : L - 1]$ . The latter stability analysis is the topic of Section 5, where we work out how the shrinking rate of the nested boxes  $\tau_\ell \times \sigma_\ell$ ,  $\ell \in [0 : L]$  and the differences  $\|c_\ell - c_{\ell+1}\|$  of two consecutive directions impact the stability of the iterated operator  $\mathfrak{I}_{\tau_L \times \sigma_L, c_L} \circ \cdots \circ \mathfrak{I}_{\tau_{\ell+1} \times \sigma_{\ell+1}, c_{\ell+1}}$ . In the interest of future reference, we formulate our findings as

$$\begin{aligned} \|k - \hat{k}_{\tau\sigma}\|_{\infty, \tau_L \times \sigma_L} &\leq \\ \sum_{\ell=0}^{L-1} \|\mathfrak{I}_{\tau_L \times \sigma_L, c_L} \circ \cdots \circ \mathfrak{I}_{\tau_{\ell+1} \times \sigma_{\ell+1}, c_{\ell+1}}\|_{\infty, \tau_L \times \sigma_L \leftarrow \tau_{\ell+1} \times \sigma_{\ell+1}} &\|k - \mathfrak{I}_{\tau_\ell \times \sigma_\ell, c_\ell}[k]\|_{\infty, \tau_{\ell+1} \times \sigma_{\ell+1}}. \end{aligned} \quad (2.17)$$

### 3. Single level analysis for the three-dimensional case

We focus here on the analysis of the 3D Helmholtz kernel, i.e., we consider  $k = g$  (with  $g_c$ ,  $\tilde{g}_{\tau\sigma}$  and  $\tilde{g}_{c, \tau\sigma}$  defined accordingly). The 2D Helmholtz kernel  $h$  can be studied using similar techniques and is discussed briefly in Section 4.

For bounding boxes  $\tau, \sigma \subseteq \mathbb{R}^n$  and a direction  $c \in \mathbb{R}^n$ , we immediately find

$$|\exp(\mathbf{i}\kappa\langle x - y, c \rangle)| = 1 \quad \text{for all } x \in \tau, y \in \sigma,$$

and we can conclude that multiplication with a plane wave does not change the maximum norm. This implies

$$\|g - \tilde{g}_{\tau\sigma}\|_{\infty, \tau \times \sigma} = \|g_c - \tilde{g}_{c, \tau\sigma}\|_{\infty, \tau \times \sigma} \quad (3.1)$$

for the approximations  $\tilde{g}_{\tau\sigma}$  and  $\tilde{g}_{c, \tau\sigma}$  defined in (2.7) and (2.6). This equation allows us to focus on interpolation error estimates for the directionally modified function  $g_c$ .



### 3.1. Tensor interpolation

The error analysis of our scheme has to gauge two sources of error: the interpolation error associated with (2.6) and the interpolation error arising from the nested interpolation (2.16). Both cases require error estimates for tensor interpolation.

Throughout the article, the *Lebesgue constant*  $\Lambda_m$  is given by

$$\Lambda_m := \|\mathfrak{I}\|_{\infty, [-1,1] \leftarrow [-1,1]}, \quad (3.2)$$

and we will make the following assumption on  $\mathfrak{I}$ :

**Assumption 3.1** *There are constants  $C_\Lambda \in \mathbb{R}_{>0}$  and  $\lambda \in \mathbb{R}_{\geq 1}$  such that*

$$\Lambda_m \leq C_\Lambda (m+1)^\lambda \quad \text{for all } m \in \mathbb{N}_0, \quad (3.3)$$

**Remark 3.2 (Chebyshev interpolation)** *A good choice for  $\mathfrak{I}$  is interpolation in the Chebyshev points  $\xi_\nu = \cos\left(\frac{2\nu+1}{2m+2}\right)$ ,  $\nu \in [0 : m]$ . In this case, [21] gives  $\Lambda_m \leq \frac{2}{\pi} \ln(m+1) + 1$ . Chebyshev interpolation satisfies Assumption 3.1 with  $C_\Lambda = \lambda = 1$ .*

We can apply tensor arguments to extend the 1D stability assumptions to the multi-dimensional case. Let us consider an axis-parallel box

$$B = [a_1, b_1] \times \cdots \times [a_n, b_n]. \quad (3.4)$$

Recall the tensor product interpolation operator  $\mathfrak{I}_B$  from (2.3) that is obtained from the 1D interpolation operator  $\mathfrak{I}$ . This operator can be written as a product of partial interpolation operators  $I \otimes \cdots \otimes \mathfrak{I}_{[a_\iota, b_\iota]} \otimes \cdots \otimes I$  that each apply interpolation only in one coordinate direction  $\iota \in [1 : n]$ . Since the stability estimate (3.2) carries over to these operators, a simple telescoping sum and the relationship between partial interpolation and one-dimensional interpolation can be used to prove the following estimate:

**Lemma 3.3 (Tensor interpolation)** *Let  $B$  be given by (3.4). For  $f \in C(B)$  define*

$$f_{x,\iota} : [-1, 1] \rightarrow \mathbb{R}, \quad t \mapsto f(x_1, \dots, x_{\iota-1}, \Phi_{[a_\iota, b_\iota]}(t), x_{\iota+1}, \dots, x_n) \quad (3.5)$$

*for all  $x \in B$  and  $\iota \in [1 : n]$ . We have*

$$\|f - \mathfrak{I}_B[f]\|_{\infty, B} \leq \sum_{\iota=1}^n \Lambda_m^{\iota-1} \max\{\|f_{x,\iota} - \mathfrak{I}[f_{x,\iota}]\|_{\infty, [a_\iota, b_\iota]} : x \in B\}.$$

In the setting of Lemma 3.3 we can find  $d, p \in \mathbb{R}^n$  such that

$$d - tp \in B, \quad f_{x,\iota}(t) = f(d - tp) \quad \text{for all } t \in [-1, 1],$$

so it suffices to bound the interpolation errors for all functions

$$f_{dp} : [-1, 1] \rightarrow \mathbb{C}, \quad t \mapsto f(d - tp),$$

with  $d, p \in \mathbb{R}^n$  such that  $d - tp \in B$  for all  $t \in [-1, 1]$ . Due to symmetry, the latter condition implies  $2\|p\| \leq \text{diam}(B)$ .

We are interested in interpolating the function  $g_c$ . Taking advantage of the fact that it only depends on the relative coordinates  $x - y$ , we obtain the following result:

**Lemma 3.4 (Univariate formulation for  $g$ )** *Let  $\epsilon \in \mathbb{R}_{\geq 0}$ , and let  $\tau, \sigma \subseteq \mathbb{R}^3$  be axis-parallel boxes. If we have*

$$\|g_{dp} - \mathfrak{I}[g_{dp}]\|_{\infty, [-1, 1]} \leq \epsilon \quad (3.6)$$

*with*

$$g_{dp}: [-1, 1] \rightarrow \mathbb{C}, \quad t \mapsto \frac{\exp(\mathbf{i}\kappa(\|d - tp\| - \langle d - tp, c \rangle))}{4\pi\|d - tp\|}, \quad (3.7)$$

*for all  $d, p \in \mathbb{R}^3$  that satisfy*

$$2\|p\| \leq \max\{\text{diam}(\tau), \text{diam}(\sigma)\}, \quad (3.8a)$$

$$d - tp \in \tau - \sigma = \{x - y : x \in \tau, y \in \sigma\} \quad \text{for all } t \in [-1, 1], \quad (3.8b)$$

*the directional tensor approximation (2.7) error is bounded by*

$$\|g - \tilde{g}_{\tau\sigma}\|_{\infty, \tau \times \sigma} \leq 6\Lambda_m^5 \epsilon. \quad (3.9)$$

*Proof.* We apply Lemma 3.3 to  $f := g_c \in C(\tau \times \sigma)$ . Given  $x \in \tau$  and  $y \in \sigma$ , the functions  $f_{(x, y), \iota}$ ,  $\iota \in [1 : 6]$ , coincide with  $g_{dp}$  for certain vectors  $d, p \in \mathbb{R}^3$  satisfying (3.8a), (3.8b). Since we have (3.6) at our disposal for *all* of these pairs of vectors, we obtain the required estimate for  $\|g_c - \tilde{g}_{c, \tau\sigma}\|_{\infty, \tau \times \sigma} = \|g - \tilde{g}_{\tau\sigma}\|_{\infty, \tau \times \sigma}$ .  $\square$

### 3.2. Holomorphic extension of $g_{dp}$

In order to obtain bounds for the interpolation error of the functions  $g_{dp}$  defined in (3.7), we consider its holomorphic extension into a neighbourhood of the interval  $[-1, 1]$ . The key step is to understand the extension of  $t \mapsto \|d - tp\|$ . In turn, the holomorphic extension of the Euclidean norm

$$\|x\| = \sqrt{\langle x, x \rangle} \quad \text{for all } x \in \mathbb{R}^n$$

requires a suitable extension of the square root, which cannot be defined in all of  $\mathbb{C}$ . We choose the principal branch given by

$$\sqrt{z} = \sqrt{|z|} \frac{z + |z|}{|z + |z||} \quad \text{for all } z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0},$$

which is holomorphic in  $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$  and maps to  $\mathbb{C}_+ := \{z \in \mathbb{C} : \Re(z) > 0\}$ . In order to identify a subset of  $\mathbb{C}$  in which  $z \mapsto \sqrt{\langle d - zp, d - zp \rangle}$  is holomorphic, we have to determine the values  $z \in \mathbb{C}$  satisfying

$$\langle d - zp, d - zp \rangle \notin \mathbb{R}_{\leq 0}.$$

**Lemma 3.5 (Extension of the Euclidean norm)** *Let  $n \in \mathbb{N}$  and  $d, p \in \mathbb{R}^n$  with  $p \neq 0$  and define*

$$w_r := \langle d, p \rangle / \|p\|^2, \quad w_i := \sqrt{\|d\|^2 / \|p\|^2 - w_r^2}, \quad w := w_r + \mathbf{i}w_i,$$

$$U_{dp} := \mathbb{C} \setminus \{w_r + \mathbf{i}y : y \in \mathbb{R}, |y| \geq w_i\}.$$

We have

$$\langle d - zp, d - zp \rangle = \|p\|^2(w - z)(\bar{w} - z) \quad \text{for all } z \in \mathbb{C}. \quad (3.10)$$

The function

$$\mathbf{n}_{dp}: U_{dp} \rightarrow \mathbb{C}_+, \quad z \mapsto \sqrt{\langle d - zp, d - zp \rangle} = \|p\| \sqrt{(w - z)(\bar{w} - z)}, \quad (3.11)$$

is well-defined and holomorphic.

*Proof.* The equality (3.10) follows from a direct computation using  $d, p \in \mathbb{R}^n$  and  $|w| = \|d\|/\|p\|$ :

$$\begin{aligned} \langle d - zp, d - zp \rangle &= \|d\|^2 - 2\langle d, p \rangle z + \|p\|^2 z^2 = \|p\|^2 |w|^2 - \|p\|^2 (w + \bar{w})z + \|p\|^2 z^2 \\ &= \|p\|^2 (w - z)(\bar{w} - z). \end{aligned}$$

In order to show that  $\mathbf{n}_{dp}$  is well-defined, it suffices to demonstrate

$$z \in U_{dp} \implies \langle d - zp, d - zp \rangle \in \mathbb{C} \setminus \mathbb{R}_{\leq 0} \quad \text{for all } z \in \mathbb{C}.$$

We use contraposition: we let  $z \in \mathbb{C}$  be such that  $\langle d - zp, d - zp \rangle \in \mathbb{R}_{\leq 0}$  and prove that this implies  $z \notin U_{dp}$ . Let  $x, y \in \mathbb{R}$  with  $z = x + \mathbf{i}y$ . We have

$$\begin{aligned} \langle d - zp, d - zp \rangle &= \|p\|^2 (w - z)(\bar{w} - z) \\ &= \|p\|^2 ((w_r - x) + \mathbf{i}(w_i - y))((w_r - x) + \mathbf{i}(-w_i - y)) \\ &= \|p\|^2 ((w_r - x)^2 - 2\mathbf{i}(w_r - x)y + w_i^2 - y^2). \end{aligned}$$

Due to  $\langle d - zp, d - zp \rangle \in \mathbb{R}_{\leq 0}$ , the imaginary part vanishes and the real part is non-positive, so  $\|p\| > 0$  yields

$$0 = 2(w_r - x)y, \quad 0 \geq (w_r - x)^2 + w_i^2 - y^2.$$

If  $y = 0$  holds, the inequality gives us  $x = w_r$  and  $w_i = 0 \leq |y|$ , i.e.,  $z \notin U_{dp}$ . Otherwise, the equation yields  $x = w_r$  and the inequality  $y^2 \geq w_i^2$ , i.e., again  $z \notin U_{dp}$ .

Since  $z \mapsto \langle d - zp, d - zp \rangle$  is holomorphic in  $U_{dp}$  and maps into the domain of the holomorphic principal square root, the composed function  $\mathbf{n}_{dp}$  is also holomorphic.  $\square$

We can only extend the norm — and therefore  $g_{dp}$  — until we reach the singularities at  $w$  and  $\bar{w}$ . In order to compute the distance between  $[-1, 1]$  and these singularities, we let  $t \in [-1, 1]$  and use (3.10) to find

$$|w - t|^2 = (w - t)(\bar{w} - t) = \frac{\langle d - tp, d - tp \rangle}{\|p\|^2} = \frac{\|d - tp\|^2}{\|p\|^2}, \quad (3.12)$$

so the distance is given by

$$\zeta := \min\{|w - t| : t \in [-1, 1]\} = \min \left\{ \frac{\|d - tp\|}{\|p\|} : t \in [-1, 1] \right\}. \quad (3.13)$$

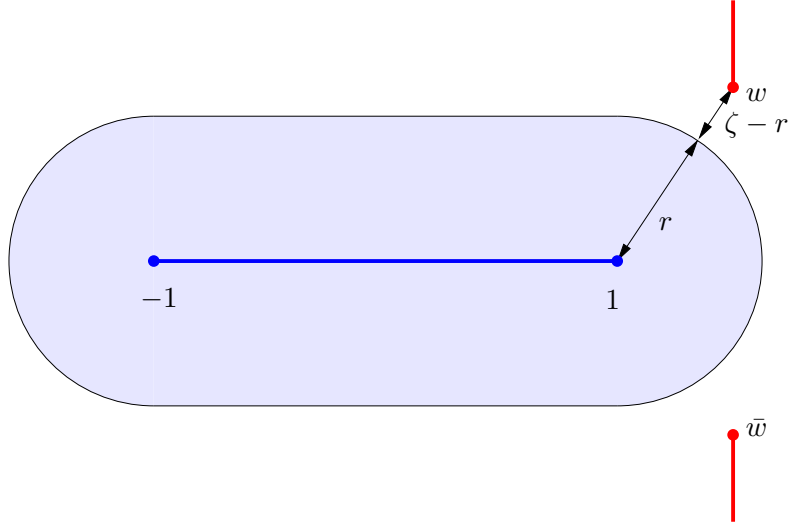


Figure 2: Domain  $U_r$  in relation to the interval  $[-1, 1]$  and  $\{w, \bar{w}\}$

**Lemma 3.6 (Bound for  $\mathbf{n}_{dp}$ )** *Let  $n \in \mathbb{N}$  and  $d, p \in \mathbb{R}^n$  with  $p \neq 0$ , and let  $\mathbf{n}_{dp}$ ,  $w$  and  $U_{dp}$  be defined as in Lemma 3.5. Let  $r \in [0, \zeta)$  and*

$$U_r := \{z \in \mathbb{C} : \exists t \in [-1, 1] : |z - t| \leq r\}. \quad (3.14)$$

*Then, we have  $U_r \subseteq U_{dp}$  and*

$$|\mathbf{n}_{dp}(z)| \geq \|p\|(\zeta - r) \quad \text{for all } z \in U_r.$$

*Proof.* We prove  $U_r \subseteq U_{dp}$  by contraposition. Let  $z \in \mathbb{C} \setminus U_{dp}$ . This implies  $z = w_r + \mathbf{i}y$  with  $y \in \mathbb{R}$  and  $y^2 \geq w_i^2$ . Due to (3.12), we have

$$|z - t|^2 = (w_r - t)^2 + y^2 \geq (w_r - t)^2 + w_i^2 = |w - t|^2 = \frac{\|d - tp\|^2}{\|p\|^2} \geq \zeta^2 > r^2$$

and therefore  $z \notin U_r$ .

Now that we have proven  $U_r \subseteq U_{dp}$ , we can consider the lower bound. Let  $z \in U_r$ . We can find  $t \in [-1, 1]$  such that  $|z - t| \leq r$ . Using (3.13), this implies

$$\begin{aligned} |w - z| &= |w - t + t - z| \geq |w - t| - |t - z| \geq \zeta - r > 0, \\ |\bar{w} - z| &= |\bar{w} - t + t - z| \geq |\bar{w} - t| - |t - z| = |w - t| - |t - z| \geq \zeta - r > 0, \end{aligned}$$

and observing

$$|\mathbf{n}_{dp}(z)| = \|p\| \sqrt{|w - z| |\bar{w} - z|} \geq \|p\|(\zeta - r)$$

completes the proof.  $\square$

We also require a bound for  $z \mapsto \exp(\mathbf{i}\kappa(\mathbf{n}_{dp}(z) - \langle d - zp, c \rangle))$ . Noting

$$|\exp(\mathbf{i}\kappa(\mathbf{n}_{dp}(z) - \langle d - zp, c \rangle))| \leq \exp(\kappa|\Im(\mathbf{n}_{dp}(z) - \langle d - zp, c \rangle)|) \quad \text{for all } z \in U_{dp},$$

our next goal is to find an upper bound for  $|\Im(\mathbf{n}_{dp}(z) - \langle d - zp, c \rangle)|$ . Following the approach of [18], we apply a Taylor expansion of  $\mathbf{n}_{dp}$  around  $t \in [-1, 1]$  to obtain the required estimate.

**Lemma 3.7** *Let  $\zeta_t \in \mathbb{R}_{>0}$  and  $r \in [0, \zeta_t)$ . We have*

$$\int_0^1 \frac{1-s}{(\zeta_t - rs)^3} ds = \frac{1}{2\zeta_t^2(\zeta_t - r)}.$$

*Proof.* The proof is straightforward for  $r = 0$ . For  $r > 0$ , the function

$$[0, 1] \rightarrow \mathbb{R}, \quad s \mapsto \frac{(1 + \zeta_t/r) - 2s}{2r(\zeta_t - rs)^2},$$

is the antiderivative of the integrand, and we can use the fundamental theorem of calculus to obtain the equation.  $\square$

**Lemma 3.8 (Exponent bound)** *Let  $d, p \in \mathbb{R}^3$  with  $p \neq 0$  and  $c \in \mathbb{R}^3$  with  $\|c\| = 1$ . Let  $r \in [0, \zeta)$ , and let  $U_r$ ,  $U_{dp}$  and  $f$  be defined as in Lemmas 3.5, 3.6. For every  $z \in U_r$ , there is a  $t \in [-1, 1]$  such that*

$$|\Im(\mathbf{n}_{dp}(z) - \langle d - zp, c \rangle)| \leq \|p\| \left( \left\| \frac{d - tp}{\|d - tp\|} - c \right\| r + \frac{1}{2(\zeta - r)} r^2 \right). \quad (3.15)$$

*Proof.* Let  $z \in U_r$ . Due to Lemma 3.6, this implies  $z \in U_{dp}$ . By definition, we can find  $t \in [-1, 1]$  with  $|z - t| \leq r$ . We have

$$\begin{aligned} \mathbf{n}_{dp}(z) &= \sqrt{\langle d - zp, d - zp \rangle} = \langle d - zp, d - zp \rangle^{1/2}, \\ \mathbf{n}'_{dp}(z) &= -\frac{\langle p, d - zp \rangle}{\langle d - zp, d - zp \rangle^{1/2}} = \frac{-\langle p, d - zp \rangle}{\mathbf{n}_{dp}(z)}, \\ \mathbf{n}''_{dp}(z) &= \frac{\langle p, p \rangle f(z) + \langle p, d - zp \rangle f'(z)}{\mathbf{n}_{dp}(z)^2} = \frac{\langle p, p \rangle f(z) - \langle p, d - zp \rangle^2 / f(z)}{\mathbf{n}_{dp}(z)^2} \\ &= \frac{\langle p, p \rangle f(z)^2 - \langle p, d - zp \rangle^2}{\mathbf{n}_{dp}(z)^3} = \frac{\langle d - zp, d - zp \rangle \langle p, p \rangle - \langle p, d - zp \rangle^2}{\mathbf{n}_{dp}(z)^3}. \end{aligned}$$

We use a Taylor expansion of  $\mathbf{n}_{dp}(z)$  around  $t$ . More precisely, with the parametrization

$$\hat{z}: [0, 1] \rightarrow U_{dp}, \quad s \mapsto \hat{z}_s := t + (z - t)s,$$

we have  $\hat{z}_0 = \hat{z}(0) = t$ ,  $\hat{z}_1 = \hat{z}(1) = z$ , and  $\hat{z}'(s) = z - t$  for all  $s \in [0, 1]$ , and the Taylor expansion of  $\mathbf{n}_{dp} \circ \hat{z}$  around  $s = 0$  yields

$$\mathbf{n}_{dp}(z) = (\mathbf{n}_{dp} \circ \hat{z})(1) = \mathbf{n}_{dp}(t) + \mathbf{n}'_{dp}(t)(z - t) + \int_0^1 \mathbf{n}''_{dp}(\hat{z}_s)(1 - s) ds (z - t)^2.$$

Hence, we obtain the equation

$$\begin{aligned}
& \mathbf{n}_{dp}(z) - \langle d - zp, c \rangle \\
&= \mathbf{n}_{dp}(t) + \mathbf{n}'_{dp}(t)(z - t) + \int_0^1 \mathbf{n}''_{dp}(\hat{z}_s)(1 - s) ds (z - t)^2 - \langle d - zp, c \rangle \\
&= \|d - tp\| - \langle d - zp, c \rangle - \frac{\langle d - tp, p \rangle}{\|d - tp\|} (z - t) \\
&\quad + \int_0^1 \frac{\langle d - \hat{z}_s p, d - \hat{z}_s p \rangle \langle p, p \rangle - \langle p, d - \hat{z}_s p \rangle^2}{\mathbf{n}_{dp}(\hat{z}_s)^3} (1 - s) ds (z - t)^2 \\
&= \left\langle d - zp, \frac{d - tp}{\|d - tp\|} - c \right\rangle \\
&\quad + \int_0^1 \frac{\langle d - \hat{z}_s p, d - \hat{z}_s p \rangle \langle p, p \rangle - \langle p, d - \hat{z}_s p \rangle^2}{\mathbf{n}_{dp}(\hat{z}_s)^3} (1 - s) ds (z - t)^2.
\end{aligned}$$

We take a closer look at the integrand. For any  $z \in \mathbb{C}$ , we find

$$\langle d - zp, d - zp \rangle \langle p, p \rangle - \langle d - zp, p \rangle^2 = \|d\|^2 \|p\|^2 - \langle d, p \rangle^2. \quad (3.16)$$

For any  $s \in [0, 1]$ , applying (3.16) twice and using the equation  $\langle d - tp, p \rangle^2 = \|d - tp\|^2 \|p\|^2 \cos^2 \angle(d - tp, p)$  yields

$$\begin{aligned}
& |\langle d - \hat{z}_s p, d - \hat{z}_s p \rangle \langle p, p \rangle - \langle d - \hat{z}_s p, p \rangle^2| = \|d\|^2 \|p\|^2 - \langle d, p \rangle^2 \\
&= \langle d - tp, d - tp \rangle \langle p, p \rangle - \langle d - tp, p \rangle^2 = \|d - tp\|^2 \|p\|^2 - \langle d - tp, p \rangle^2 \\
&= \|d - tp\|^2 \|p\|^2 \sin^2 \angle(d - tp, p) \leq \|d - tp\|^2 \|p\|^2.
\end{aligned}$$

We have proven

$$\begin{aligned}
& \mathbf{n}_{dp}(z) - \langle d - zp, c \rangle \\
&= \left\langle d - zp, \frac{d - tp}{\|d - tp\|} - c \right\rangle + \int_0^1 \frac{\|d - tp\|^2 \|p\|^2 \sin^2 \angle(d - tp, p)}{\mathbf{n}_{dp}(\hat{z}_s)^3} (1 - s) ds (z - t)^2.
\end{aligned} \quad (3.17)$$

Using (3.10), we have

$$\frac{\|d - tp\|^2 \|p\|^2}{|\mathbf{n}_{dp}(\hat{z}_s)^3|} = \frac{\|p\|^4 |w - t|^2}{\|p\|^3 |w - \hat{z}_s|^3} \leq \frac{\|p\| |w - t|^2}{(|w - t| - |z - t|s)^3} \leq \frac{\|p\| |w - t|^2}{(|w - t| - rs)^3},$$

and applying Lemma 3.7 to  $\zeta_t := |w - t|$  yields

$$\begin{aligned}
& \left| \int_0^1 \frac{\|d - tp\|^2 \|p\|^2 \sin^2 \angle(d - tp, p)}{\mathbf{n}_{dp}(\hat{z}_s)^3} (1 - s) ds (z - t)^2 \right| \\
& \leq \|p\| |w - t|^2 \int_0^1 \frac{1 - s}{(\zeta_t - rs)^3} ds |z - t|^2 = \frac{\|p\| |w - t|^2}{2\zeta_t^2 (\zeta_t - r)} |z - t|^2 \leq \frac{\|p\|}{2(\zeta_t - r)} r^2.
\end{aligned}$$

We write  $z = x + \mathbf{i}y$  with  $x, y \in \mathbb{R}$ . This implies  $z - t = (x - t) + \mathbf{i}y$  and  $|z - t| \geq |y|$ , so we can apply the Cauchy-Schwarz inequality to get

$$\begin{aligned} \left| \Im \left( \left\langle d - zp, \frac{d - tp}{\|d - tp\|} - c \right\rangle \right) \right| &= \left| \left\langle yp, \frac{d - tp}{\|d - tp\|} - c \right\rangle \right| \leq |y| \|p\| \left\| \frac{d - tp}{\|d - tp\|} - c \right\| \\ &\leq \|p\| \left\| \frac{d - tp}{\|d - tp\|} - c \right\| |z - t| \leq \|p\| \left\| \frac{d - tp}{\|d - tp\|} - c \right\| r. \end{aligned}$$

Combining both estimates with (3.17) and  $\zeta_t \geq \zeta$  gives us (3.15).  $\square$

### 3.3. Admissibility conditions

In order to obtain useful estimates for the interpolation error of  $g_{dp}$ , we have to control the absolute value of the holomorphic extension

$$g_{dp}: U_r \rightarrow \mathbb{C}, \quad z \mapsto \frac{\exp(\mathbf{i}\kappa(\mathbf{n}_{dp}(z) - \langle d - zp, c \rangle))}{4\pi\mathbf{n}_{dp}(z)},$$

of  $g_{dp}$  (cf. (3.7)). For the denominator,  $d - tp \in \tau - \sigma$  implies that we have to ensure that  $\tau$  and  $\sigma$  are well-separated. This is guaranteed by the *standard admissibility condition*

$$\max\{\text{diam}(\tau), \text{diam}(\sigma)\} \leq \eta_2 \text{dist}(\tau, \sigma), \quad (3.18a)$$

where  $\eta_2 \in \mathbb{R}_{>0}$  is a parameter that can be chosen to balance the computational complexity and the speed of convergence.

For the numerator, we have

$$|\exp(\mathbf{i}\kappa(\mathbf{n}_{dp}(z) - \langle d - zp, c \rangle))| \leq \exp(\kappa|\Im(\mathbf{n}_{dp}(z) - \langle d - zp, c \rangle)|),$$

and Lemma 3.8 suggests that we should find a bound  $C \in \mathbb{R}_{\geq 0}$  satisfying

$$\kappa\|p\| \frac{r}{2(\zeta - r)} \leq C, \quad \kappa\|p\| \left\| \frac{d - tp}{\|d - tp\|} - c \right\| \leq C \quad \text{for all } t \in [-1, 1],$$

with a suitable  $r \in [0, \zeta)$ . We first consider the second inequality involving the direction  $c$ . Instead of looking for a bound for all  $t \in [-1, 1]$ , we only consider the centers  $m_\tau \in \tau$  and  $m_\sigma \in \sigma$  of the two boxes and require

$$\kappa \left\| \frac{m_\tau - m_\sigma}{\|m_\tau - m_\sigma\|} - c \right\| \leq \frac{\eta_1}{\max\{\text{diam}(\tau), \text{diam}(\sigma)\}} \quad (3.18b)$$

for a second admissibility parameter  $\eta_1 \in \mathbb{R}_{>0}$ . In order to obtain the required estimate, we have to ensure that  $m_\tau - m_\sigma$  is sufficiently close to  $d - tp$  by using the condition

$$\kappa \max\{\text{diam}^2(\tau), \text{diam}^2(\sigma)\} \leq \eta_2 \text{dist}(\tau, \sigma). \quad (3.18c)$$

**Lemma 3.9 (Approximate directions)** *Assume that  $\tau, \sigma$ , and  $c$  satisfy the conditions (3.18b) and (3.18c). Let  $d, p \in \mathbb{R}^n$  be vectors satisfying (3.8a) and (3.8b). Then we have*

$$\left\| \frac{d - tp}{\|d - tp\|} - c \right\| \leq \frac{\eta_1 + \eta_2}{2\kappa\|p\|} \quad \text{for all } t \in [-1, 1].$$

*Proof.* Let  $t \in [-1, 1]$  and  $q := \max\{\text{diam}(\tau), \text{diam}(\sigma)\}$ . (3.8b) and (3.18c) yield

$$\|d - tp\| \geq \text{dist}(\tau, \sigma) \geq \frac{\kappa}{\eta_2} q^2, \quad \left\| (d - tp) \frac{\eta_2}{\kappa q^2} \right\| \geq 1.$$

Due to  $m_\tau \in \tau$  and  $m_\sigma \in \sigma$ , we can apply (3.18c) to find

$$\|m_\tau - m_\sigma\| \geq \text{dist}(\tau, \sigma) \geq \frac{\kappa}{\eta_2} q^2, \quad \left\| (m_\tau - m_\sigma) \frac{\eta_2}{\kappa q^2} \right\| \geq 1.$$

Since projecting two points outside the unit ball to its surface does not increase their distance (cf. [3, Lemma 7]), we obtain

$$\left\| \frac{d - tp}{\|d - tp\|} - \frac{m_\tau - m_\sigma}{\|m_\tau - m_\sigma\|} \right\| \leq \|(d - tp) - (m_\tau - m_\sigma)\| \frac{\eta_2}{\kappa q^2}.$$

Due to (3.8b), we can find  $x \in \tau$  and  $y \in \sigma$  with  $d - tp = x - y$  and obtain

$$\|(d - tp) - (m_\tau - m_\sigma)\| = \|(x - m_\tau) - (y - m_\sigma)\| \leq q/2 + q/2 = q.$$

Combining the estimates with (3.18b) yields

$$\begin{aligned} \left\| \frac{d - tp}{\|d - tp\|} - c \right\| &\leq \left\| \frac{d - tp}{\|d - tp\|} - \frac{m_\tau - m_\sigma}{\|m_\tau - m_\sigma\|} \right\| + \left\| \frac{m_\tau - m_\sigma}{\|m_\tau - m_\sigma\|} - c \right\| \\ &\leq \|(d - tp) - (m_\tau - m_\sigma)\| \frac{\eta_2}{\kappa q^2} + \frac{\eta_1}{\kappa q} \leq \frac{\eta_2 + \eta_1}{\kappa q}. \end{aligned}$$

To complete the proof, we recall that (3.8a) yields  $2\|p\| \leq q$ .  $\square$

The condition (3.18b) is trivially satisfied by  $c = 0$  if  $\kappa \max\{\text{diam}(\tau), \text{diam}(\sigma)\} \leq \eta_1$  holds, i.e., if we are in the low-frequency setting. Choosing  $c = 0$  is particularly attractive, since it means that we use standard polynomial interpolation. We require

$$\kappa \max\{\text{diam}(\tau), \text{diam}(\sigma)\} \leq \eta_1 \implies c = 0. \quad (3.19)$$

We collect our findings in the following definition.

**Definition 3.10 (Parabolic admissibility)** *A triple  $(\tau, \sigma, c)$  satisfies the parabolic admissibility condition if the three conditions (3.18) together with (3.19) hold.*



### 3.4. Interpolation error

The result of Lemma 3.8 provides us with an upper bound for the exponential term in the numerator of the definition (3.7) of  $g_{dp}$ , while Lemma 3.6 provides us with a lower bound for the denominator. Since we have assumed the interpolation scheme to be stable, we only have to prove that a polynomial approximation of  $g_{dp}$  exists. Following [13, Chapter 7], the existence of a holomorphic extension in a *Bernstein elliptic disc*

$$\mathcal{D}_\varrho := \left\{ z = x + iy : x, y \in \mathbb{R}, \left( \frac{2x}{\varrho + 1/\varrho} \right)^2 + \left( \frac{2y}{\varrho - 1/\varrho} \right)^2 < 1 \right\} \quad \text{for all } \varrho \in \mathbb{R}_{>1} \quad (3.20)$$

already implies the existence of such an approximation.

**Lemma 3.11 (polynomial approximability)** *Let  $\hat{\varrho} \in \mathbb{R}_{>1}$ , and let  $f : \mathcal{D}_{\hat{\varrho}} \rightarrow \mathbb{C}$  be holomorphic. Given  $\varrho \in (1, \hat{\varrho})$  and  $m \in \mathbb{N}$ , there is a polynomial  $\pi \in \Pi_m$  of degree  $m$  such that*

$$\|f - \pi\|_{\infty, [-1,1]} \leq \frac{2}{\varrho - 1} \varrho^{-m} \max\{|f(z)| : z \in \overline{\mathcal{D}_\varrho}\}.$$

*Proof.* This is [13, eqn. (8.7), Chap. 7].  $\square$

Our Lemmas 3.6 and 3.8 can be used to obtain bounds for the holomorphic extension of  $g_{dp}$  in the domains  $U_r$ . In order to apply Lemma 3.11, we simply have to find  $\varrho > 1$  such that  $\overline{\mathcal{D}_\varrho} \subseteq U_r$ .

**Lemma 3.12 (Inclusion)** *Let  $r \in \mathbb{R}_{>0}$ , and let  $\varrho := \sqrt{r^2 + 1} + r$ . We have  $\overline{\mathcal{D}_\varrho} \subseteq U_r$ .*

*Proof.* This is [5, Lemma 4.77].  $\square$

**Theorem 3.13 (Approximation of  $g_{dp}$ )** *Let  $c \in \mathbb{R}^3$  and axis-parallel boxes  $\tau, \sigma$  satisfy the admissibility conditions (3.18). Let  $d, p \in \mathbb{R}^3$  satisfy (3.8a) and (3.8b). Let*

$$\hat{\varrho} := \min \left\{ 2, \frac{3}{2\eta_2} + 1 \right\}. \quad (3.21)$$

*There are constants  $C_{\text{in}} \in \mathbb{R}_{\geq 0}$  and  $\alpha \in \mathbb{R}_{>1}$  depending only on the admissibility parameters  $\eta_1, \eta_2$  and the stability constants  $C_\Lambda, \lambda$  (cf. (3.3)) such that*

$$\|g_{dp} - \mathcal{I}[g_{dp}]\|_{\infty, [-1,1]} \leq \frac{C_{\text{in}}}{4\pi \text{dist}(\tau, \sigma)} \alpha^{-m/2} \hat{\varrho}^{-m} \quad \text{for all } m \in \mathbb{N}.$$

*Proof.* Due to (3.8b), we have

$$\|d - tp\| \geq \text{dist}(\tau, \sigma) \quad \text{for all } t \in [-1, 1],$$

and therefore

$$\zeta = \min \left\{ \frac{\|d - tp\|}{\|p\|} : t \in [-1, 1] \right\} \geq \frac{\text{dist}(\tau, \sigma)}{\|p\|}.$$

Combining (3.8a) with the standard admissibility condition (3.18a) and the parabolic admissibility condition (3.18c), we obtain  $\zeta \geq 2/\eta_2$  and  $\zeta \geq 4\kappa\|p\|/\eta_2$ . For the Taylor expansion, we choose the radius

$$r := \min \left\{ 1, \frac{3}{4}\zeta \right\} \geq \hat{r} := \min \left\{ 1, \frac{3}{2\eta_2} \right\}$$

and consider  $z \in U_r$ . Lemma 3.9 yields

$$\left\| \frac{d - tp}{\|d - tp\|} - c \right\| \leq \frac{\eta_1 + \eta_2}{2\kappa\|p\|} \quad \text{for all } t \in [-1, 1],$$

and (3.15) takes the form

$$\begin{aligned} |\Im(\mathbf{n}_{dp}(z) - \langle d - zp, c \rangle)| &\leq \|p\| \left( \left\| \frac{d - tp}{\|d - tp\|} - c \right\| r + \frac{1}{2(\zeta - r)} r^2 \right) \\ &\leq \|p\| \left( \frac{\eta_1 + \eta_2}{2\kappa\|p\|} + \frac{\eta_2}{8\kappa\|p\|(1 - r/\zeta)} \right) \\ &= \frac{1}{\kappa} \left( \frac{\eta_1 + \eta_2}{2} + \frac{\eta_2}{2} \right) \leq \frac{\eta_1 + \eta_2}{\kappa}. \end{aligned}$$

For the denominator of  $g_{dp}$ , we use Lemma 3.6 to find

$$|\mathbf{n}_{dp}(z)| \geq \|p\|(\zeta - r) \geq \text{dist}(\tau, \sigma)(1 - r/\zeta) \geq \text{dist}(\tau, \sigma)/4$$

and arrive at

$$|g_{dp}(z)| \leq \frac{|\exp(\mathbf{i}\kappa(\mathbf{n}_{dp}(z) - \langle d - zp, c \rangle))|}{4\pi|f(z)|} \leq \frac{\exp(\eta_1 + \eta_2)}{\pi \text{dist}(\tau, \sigma)}.$$

According to Lemma 3.12,  $U_r$  contains  $\overline{\mathcal{D}}_\varrho$  for

$$\varrho = \sqrt{r^2 + 1} + r \geq \sqrt{\hat{r}^2 + 1} + \hat{r} > \hat{r} + 1 = \min \left\{ 2, \frac{3}{2\eta_2} + 1 \right\} = \hat{\varrho}.$$

Let  $\alpha := (\sqrt{\hat{r}^2 + 1} + \hat{r})/(\hat{r} + 1)$ . We have  $\varrho \geq \alpha\hat{\varrho}$ ,  $\alpha > 1$ , and (3.3) yields that

$$C_{\text{in}} := \sup \left\{ \frac{8(\Lambda_m + 1)}{(\hat{\varrho} - 1)\alpha^{m/2}} \exp(\eta_1 + \eta_2) : m \in \mathbb{N} \right\}$$

is finite. Let now  $m \in \mathbb{N}$ . Lemma 3.11 gives us  $\pi \in \Pi_m$  with

$$\begin{aligned} \|g_{dp} - \pi\|_{\infty, [-1, 1]} &\leq \frac{2}{\hat{\varrho} - 1} \varrho^{-m} \max\{|f(z)| : z \in \overline{\mathcal{D}}_\varrho\} \\ &\leq \frac{2}{(\hat{\varrho} - 1)\alpha^{m/2}} \alpha^{-m/2} \hat{\varrho}^{-m} \frac{4\exp(\eta_1 + \eta_2)}{4\pi \text{dist}(\tau, \sigma)}. \end{aligned}$$

Using the well-known best-approximation property of stable interpolation schemes, we obtain

$$\|g_{dp} - \mathcal{I}[g_{dp}]\|_{\infty, [-1, 1]} \leq (1 + \Lambda_m) \|g_{dp} - \pi\|_{\infty, [-1, 1]} \leq \frac{C_{\text{in}}}{4\pi \text{dist}(\tau, \sigma)} \alpha^{-m/2} \hat{\varrho}^{-m},$$

and this is our final result.  $\square$

**Corollary 3.14 (Interpolation error for  $g$ )** *Let  $c \in \mathbb{R}^3$  and axis-parallel boxes  $\tau, \sigma$  satisfy the admissibility conditions (3.18). Let  $\hat{\varrho}$  be given by (3.21). Then there is a constant  $C_{\text{mi}} \in \mathbb{R}_{\geq 0}$  depending only on the admissibility parameters  $\eta_1, \eta_2$  and the stability constants  $C_\Lambda, \lambda$  (cf. (3.3)) such that*

$$\|g - \tilde{g}_{\tau\sigma}\|_{\infty, \tau \times \sigma} \leq \frac{C_{\text{mi}}}{4\pi \text{dist}(\tau, \sigma)} \hat{\varrho}^{-m} \quad \text{for all } m \in \mathbb{N}.$$

*Proof.* Let  $m \in \mathbb{N}$ . We combine Lemma 3.4 with Theorem 3.13 to obtain

$$\|g - \tilde{g}_{\tau\sigma}\|_{\infty, \tau \times \sigma} \leq 6\Lambda_m^5 \frac{C_{\text{in}}}{4\pi \text{dist}(\tau, \sigma)} \alpha^{-m/2} \hat{\varrho}^{-m}$$

with  $\alpha > 1$ . Due to the stability assumption (3.3), the supremum

$$C_{\text{mi}} := \sup \left\{ \frac{6\Lambda_m^5 C_{\text{in}}}{\alpha^{m/2}} : m \in \mathbb{N} \right\}$$

is finite and we conclude

$$\|g - \tilde{g}_{\tau\sigma}\|_{\infty, \tau \times \sigma} \leq \frac{C_{\text{mi}}}{4\pi \text{dist}(\tau, \sigma)} \hat{\varrho}^{-m}.$$

□

**Remark 3.15 (Asymptotic rate)** *According to [13, Theorem 8.1, Chap. 7], we can expect the error to be bounded by  $C_r \varrho^{-m}$  with  $\varrho = \sqrt{r^2 + 1} + r$  for any  $r < \zeta$ . However,  $C_r \rightarrow \infty$  for  $r \rightarrow \zeta$  is possible.*

*In our proof of Theorem 3.13, we have chosen  $r$  in a way that leads to a particularly simple estimate for the exponential term.*

## 4. Two-dimensional Helmholtz kernel

The core of the analysis of the 3D case in Section 3 is the detailed analysis of the holomorphic extension of the Euclidean norm, i.e., the function  $\mathbf{n}_{dp}$ , as it allows for good control of the functions  $z \mapsto \exp(\mathbf{i}\kappa(\mathbf{n}_{dp}(z) - \langle d - zp, c \rangle))$ . This opens the door to the analysis of more general kernel functions  $k$  for which (the holomorphic extension of) the “non-oscillatory” part  $k(x, y) \exp(-\mathbf{i}\kappa\|x - y\|)$  can be controlled. A particularly interesting case are translation invariant kernel functions of the form  $k(x, y) = k_1(\kappa, \|x - y\|)$ , where the map  $z \mapsto k_1(\kappa, z)$  is holomorphic on  $\mathbb{C}_+$  and satisfies suitable conditions there. The two-dimensional Helmholtz kernel  $h$  is such an example.

**Lemma 4.1** *There exists a  $C > 0$  such that*

$$\left| H_0^{(1)}(z) \exp(-\mathbf{i}z) \right| \leq C \min \left\{ 1 + |\ln |z||, 1/\sqrt{|z|} \right\} \quad \text{for all } z \in \mathbb{C}_+.$$

*Proof.* The bound is obtained by studying the cases of small  $|z|$  and large  $|z|$  separately. For small  $z$  we use the fact that  $H_0^{(1)}(z) = J_0(z) + \mathbf{i}Y_0(z)$  and that  $J_0$  is analytic with  $\lim_{z \rightarrow 0} J_0(z) = 1$  and  $Y_0(z) \sim 2/\pi (\ln(z/2) + \gamma)$  (as  $z \rightarrow 0$ ,  $z \in \mathbb{C}_+$ ) with Euler-Mascheroni's constant  $\gamma$  (cf. [1, (9.1.12), (9.1.13)]) so that for any  $R > 0$  one has a  $C_R > 0$  such that

$$\left| H_0^{(1)}(z) \exp(-\mathbf{i}\zeta) \right| \leq C_R(1 + |\ln|z||) \quad \text{for all } z \in B_R(0) \cap \mathbb{C}_+.$$

For large  $z$ , one uses [20, Chap. 7, eqns. (13.2), (13.3)] with  $n = 1$  to get

$$\left| H_0^{(1)}(z) \exp(-\mathbf{i}(z - \pi/4)) \right| \leq \sqrt{\frac{2}{\pi|z|}} \left( 1 + \frac{\pi}{8|z|} \exp\left(\frac{\pi}{8|z|}\right) \right).$$

□

Using Lemma 4.1 it is possible to formulate the approximation result corresponding to Theorem 3.13.

**Lemma 4.2** *Let  $c \in \mathbb{R}^2$  and axis-parallel boxes  $\tau, \sigma$  satisfy the admissibility conditions (3.18). Let  $d, p \in \mathbb{R}^2$  be vectors satisfying (3.8a) and (3.8b). Let  $\hat{\varrho}$  be given by (3.21). Define*

$$h_{dp}: [-1, 1] \rightarrow \mathbb{C}, \quad t \mapsto \frac{\mathbf{i}}{4} H_0^{(1)}(\kappa \|d - tp\|) \exp(-\mathbf{i}\kappa \langle d - tp, c \rangle).$$

*Then there are constants  $C_{\text{in}} \in \mathbb{R}_{\geq 0}$  and  $\alpha \in \mathbb{R}_{>1}$  depending only on the admissibility parameters  $\eta_1, \eta_2$  and the stability constants  $C_\Lambda, \lambda$  (cf. (3.3)) such that for all  $m \in \mathbb{N}$*

$$\|h_{dp} - \mathfrak{I}[h_{dp}]\|_{\infty, [-1, 1]} \leq C_{\text{in}} \min\{1 + |\ln(\kappa \text{dist}(\tau, \sigma))|, (\kappa \text{dist}(\tau, \sigma))^{-1/2}\} \alpha^{-m/2} \hat{\varrho}^{-m}.$$

*Proof.* The key is to recall that  $z \mapsto \mathbf{n}_{dp}(z)$  is the holomorphic extension of  $t \mapsto \|d - tp\|$  so that the analog of the univariate function  $g_{dp}$  reads

$$h_{dp}(z) = \frac{\mathbf{i}}{4} \underbrace{H_0^{(1)}(\kappa \mathbf{n}_{dp}(z)) \exp(-\mathbf{i}\kappa \mathbf{n}_{dp}(z))}_{=: A(z)} \underbrace{\exp(\mathbf{i}\kappa (\mathbf{n}_{dp}(z) - \langle d - zp, c \rangle))}_{=: B(z)}.$$

Following the proof of Theorem 3.13 we have to control  $h_{dp}$  on  $U_r$  (with  $r$  given in the proof of Theorem 3.13). By the proof of Theorem 3.13 we have for  $z \in U_r$  that  $\mathbf{n}_{dp}(z) \in \mathbb{C}_+$  and that  $|\mathbf{n}_{dp}(z)| \geq \text{dist}(\tau, \sigma)/4$ . We conclude with Lemma 4.1 that

$$|A(z)| \leq C \min\{1 + |\ln(\kappa \text{dist}(\sigma, \tau))|, (\kappa \text{dist}(\sigma, \tau))^{-1/2}\}.$$

The term  $B(z)$  is estimated in the proof of Theorem 3.13 by  $|B(z)| \leq \exp(\eta_1 + \eta_2)$ . The result follows as in Theorem 3.13. □

Reasoning as in the proof of Corollary 3.14 we arrive at the following result.

**Corollary 4.3 (Interpolation error for  $h$ )** *Let  $c \in \mathbb{R}^2$  with  $\|c\| = 1$  and axis-parallel boxes  $\tau, \sigma$  satisfy the admissibility conditions (3.18). Let  $\hat{\varrho}$  be given by (3.21). Then there is a constant  $C_{\text{mi}} \in \mathbb{R}_{\geq 0}$  depending only on the admissibility parameters  $\eta_1, \eta_2$  and the stability constants  $C_\Lambda, \lambda$  (cf. (3.3)) such that for all  $m \in \mathbb{N}$*

$$\|h - \tilde{h}_{\tau\sigma}\|_{\infty, \tau \times \sigma} \leq C_{\text{mi}} \min\{1 + |\ln(\kappa \text{dist}(\tau, \sigma))|, (\kappa \text{dist}(\tau, \sigma))^{-1/2}\} \hat{\varrho}^{-m}.$$

## 5. Nested approximation

As mentioned before, the crux of polylogarithmic-linear complexity algorithms is a nested multilevel structure. The vital ingredient that permits this structure is the approximation step (2.16). In this section, we analyze the impact of this step. Structurally similar analyses can be found in [8, 23].

### 5.1. Reduction to univariate nested interpolation

We recall the setting of Section 2.2.3: we are given sequences of axis-parallel boxes

$$\tau_0 \supseteq \tau_1 \supseteq \cdots \supseteq \tau_L, \quad \sigma_0 \supseteq \sigma_1 \supseteq \cdots \supseteq \sigma_L \quad (5.1)$$

and a sequence  $c_0, \dots, c_L \in \mathbb{R}^n$  of directions. We are interested in the directional interpolation operators  $\mathfrak{J}_{\tau_\ell, c_\ell}$  given by

$$\mathfrak{J}_{\tau_\ell, c_\ell}[u] = \exp(\mathbf{i}\kappa\langle c_\ell, \cdot \rangle) \mathfrak{J}_{\tau_\ell}[\exp(-\mathbf{i}\kappa\langle c_\ell, \cdot \rangle)u] \quad \text{for all } u \in C(\tau_\ell), \quad (5.2)$$

and similar operators  $\mathfrak{J}_{\sigma_\ell, -c_\ell}$  for the source clusters. With the aid of these operators, we write the operators  $\mathfrak{J}_{\tau_\ell \times \sigma_\ell, c_\ell}$  of (2.15) as tensor product operators

$$\mathfrak{J}_{\tau_\ell \times \sigma_\ell, c_\ell} = \mathfrak{J}_{\tau_\ell, c_\ell} \otimes \mathfrak{J}_{\sigma_\ell, -c_\ell} \quad \text{for all } \ell \in [0 : L]$$

and approximate the kernel function  $k$  by

$$\tilde{k}_{\tau\sigma} = \mathfrak{J}_{\tau_L \times \sigma_L, c_L} \circ \cdots \circ \mathfrak{J}_{\tau_0 \times \sigma_0, c_0}[k].$$

In order to estimate the approximation error, we can rely on (2.17) to find that we only need a stability estimate for the iterated operators, since we already have Corollaries 3.14 and 4.3 at our disposal. The iterated operators can be rewritten as

$$\begin{aligned} \mathfrak{J}_{\tau_L \times \sigma_L, c_L} \circ \cdots \circ \mathfrak{J}_{\tau_{\ell+1} \times \sigma_{\ell+1}, c_{\ell+1}} &= \left( \mathfrak{J}_{\tau_L, c_L} \circ \cdots \circ \mathfrak{J}_{\tau_{\ell+1}, c_{\ell+1}} \right) \\ &\quad \otimes \left( \mathfrak{J}_{\sigma_L, -c_L} \circ \cdots \circ \mathfrak{J}_{\sigma_{\ell+1}, -c_{\ell+1}} \right). \end{aligned}$$

Since

$$\begin{aligned} &\|\mathfrak{J}_{\tau_L \times \sigma_L, c_L} \circ \cdots \circ \mathfrak{J}_{\tau_{\ell+1} \times \sigma_{\ell+1}, c_{\ell+1}}\|_{\infty, \tau_L \times \sigma_L \leftarrow \tau_{\ell+1} \times \sigma_{\ell+1}} \\ &\leq \|\mathfrak{J}_{\tau_L, c_L} \circ \cdots \circ \mathfrak{J}_{\tau_{\ell+1}, c_{\ell+1}}\|_{\infty, \tau_L \leftarrow \tau_{\ell+1}} \|\mathfrak{J}_{\sigma_L, -c_L} \circ \cdots \circ \mathfrak{J}_{\sigma_{\ell+1}, -c_{\ell+1}}\|_{\infty, \sigma_L \leftarrow \sigma_{\ell+1}}, \end{aligned} \quad (5.3)$$

we have reduced the quest for the stability estimates required by (2.17) to a stability analysis of the operators  $\mathfrak{J}_{\tau_L, c_L} \circ \cdots \circ \mathfrak{J}_{\tau_{\ell+1}, c_{\ell+1}}$  and  $\mathfrak{J}_{\sigma_L, -c_L} \circ \cdots \circ \mathfrak{J}_{\sigma_{\ell+1}, -c_{\ell+1}}$ . Their stability properties depend on how quickly the boxes shrink and how small the differences  $\|c_{\ell+1} - c_\ell\|$  are. Our final result is recorded in Theorem 5.7.

**Remark 5.1** We have  $\mathfrak{J}_{\tau_\ell, 0} = \mathfrak{J}_{\tau_\ell}$  and  $\mathfrak{J}_{\sigma_\ell, 0} = \mathfrak{J}_{\sigma_\ell}$ .

If  $u \in C(\sigma_\ell)$  is real-valued, we have  $\mathfrak{J}_{\sigma_\ell, -c_\ell}[u] = \overline{\mathfrak{J}_{\sigma_\ell, c_\ell}[u]}$ .

**Remark 5.2 (Re-interpolated Lagrange polynomials)** *Due to Section 2.2.3, we can also reduce the error analysis to estimating  $L_{\tau c, \nu} - \bar{L}_{\tau c, \nu}$ . We expect the latter approach to be slightly sharper than resorting to the bounds (2.17) and (5.3).*

Since the operators  $\mathfrak{J}_{\tau_\ell, c_\ell}$  have product structure, their analysis can be broken down further to that of understanding operators acting on univariate functions. Specifically, writing the axis-parallel boxes  $\tau_\ell$  in the form

$$\tau_\ell = [a_{\ell,1}, b_{\ell,1}] \times \cdots \times [a_{\ell,n}, b_{\ell,n}],$$

and observing  $\exp(\mathbf{i}\kappa\langle c, x \rangle) = \exp(\mathbf{i}\kappa c_1 x_1) \cdots \exp(\mathbf{i}\kappa c_n x_n)$ , we have

$$\begin{aligned} \mathfrak{J}_{\tau_\ell, c_\ell}[u] &= \exp(\mathbf{i}\kappa\langle c_\ell, \cdot \rangle) \mathfrak{J}_{[a_{\ell,1}, b_{\ell,1}]} \otimes \cdots \otimes \mathfrak{J}_{[a_{\ell,n}, b_{\ell,n}]} [\exp(-\mathbf{i}\kappa\langle c_\ell, \cdot \rangle)u] \\ &= \mathfrak{J}_{\ell,1} \otimes \cdots \otimes \mathfrak{J}_{\ell,n}[u] \quad \text{for all } u \in C(\tau_\ell) \end{aligned}$$

with the one-dimensional interpolation operators

$$\mathfrak{J}_{\ell, \iota} : C[a_{\ell, \iota}, b_{\ell, \iota}] \rightarrow C[a_{\ell, \iota}, b_{\ell, \iota}], \quad u \mapsto \exp(\mathbf{i}\kappa c_{\ell, \iota} \cdot) \mathfrak{J}_{[a_{\ell, \iota}, b_{\ell, \iota}]} [\exp(-\mathbf{i}\kappa c_{\ell, \iota} \cdot)u] \quad (5.4)$$

for all  $\ell \in [0 : L]$  and  $\iota \in [1 : n]$ . We note

$$\mathfrak{J}_{\tau_L, c_L} \circ \cdots \circ \mathfrak{J}_{\tau_{\ell+1}, c_{\ell+1}} = (\mathfrak{J}_{L,1} \circ \cdots \mathfrak{J}_{\ell+1,1}) \otimes \cdots \otimes (\mathfrak{J}_{L,n} \circ \cdots \mathfrak{J}_{\ell+1,n}), \quad (5.5)$$

so that the stability analysis of  $\mathfrak{J}_{\tau_L, c_L} \circ \cdots \circ \mathfrak{J}_{\tau_{\ell+1}, c_{\ell+1}}$  is indeed reduced to that of the operators  $(\mathfrak{J}_{L, \iota} \circ \cdots \mathfrak{J}_{\ell+1, \iota})$  for  $\iota \in [1 : n]$ .

## 5.2. Recursive reinterpolation in 1D

Let  $\mathfrak{C} := (J_\ell)_{\ell=0}^L$  be a tuple of non-empty intervals

$$J_0 \supseteq J_1 \supseteq J_2 \supseteq \cdots \supseteq J_L.$$

We assume that there is a *contraction factor*  $\bar{q} \in \mathbb{R}$  such that

$$\frac{|J_\ell|}{|J_{\ell-1}|} \leq \bar{q} < 1 \quad \text{for all } \ell \in [1 : L]. \quad (5.6)$$

We fix  $c_0, \dots, c_L \in \mathbb{R}$  and denote the weighted interpolation operators by

$$\mathfrak{J}_\ell : C(J_\ell) \rightarrow C(J_\ell), \quad u \mapsto \exp(\mathbf{i}\kappa c_\ell \cdot) \mathfrak{J}_{J_\ell} [\exp(-\mathbf{i}\kappa c_\ell \cdot)u] \quad \text{for all } \ell \in [0 : L].$$

The iterated interpolation operator is given by

$$\mathfrak{J}_{\mathfrak{C}} := \mathfrak{J}_L \circ \cdots \circ \mathfrak{J}_0.$$

The proof uses the Bernstein estimate to bound a polynomial in a Bernstein disc  $\mathcal{D}_\alpha$  and then applies Lemma 3.11 to find an approximation in a sub-interval. For the second step, we require the following geometrical result.

**Lemma 5.3 (Inclusion)** *Let  $-1 \leq a < b \leq 1$  and  $h := (b - a)/2$ . For  $\alpha > 1$  denote by*

$$\mathcal{D}_\alpha^{a,b} := \Phi_{[a,b]}(\mathcal{D}_\alpha).$$

*the transformed Bernstein disc  $\mathcal{D}_\alpha$  (cf. (3.20)) for the interval  $[a, b]$ . Fix  $\varepsilon \in (0, 1)$ . Then there is a  $\varrho_0 > 1$  (depending solely on  $\varepsilon$ ) such that*

$$\mathcal{D}_{(1-\varepsilon)\varrho/h}^{a,b} \subset \mathcal{D}_\varrho \quad \text{for all } \varrho \geq \varrho_0.$$

*Proof.* We exploit that for large  $\varrho$  the Bernstein disc  $\mathcal{D}_\varrho$  is essentially a (classical) disc of radius  $\varrho/2$ . We start from the following inclusion of discs in Bernstein elliptic discs and *vice versa*:

$$B_{(\varrho-1/\varrho)/2}(0) \subset \mathcal{D}_\varrho \subset B_{(\varrho+1/\varrho)/2}(0),$$

where  $B_r(x) = \{|z - x| \leq r : z \in \mathbb{C}\}$  denotes the closed disc around  $x$  of radius  $r$ . Hence, we have to show (for  $\varrho$  sufficiently large) that for  $\alpha = (1 - \varepsilon)\varrho/h$  we have

$$\begin{aligned} \mathcal{D}_\alpha^{a,b} &= \frac{a+b}{2} + h\mathcal{D}_\alpha \subset \frac{a+b}{2} + hB_{(\alpha+1/\alpha)/2}(0) \\ &= B_{h(\alpha+1/\alpha)/2}\left(\frac{a+b}{2}\right) \stackrel{!}{\subset} B_{(\varrho-1/\varrho)/2}(0); \end{aligned}$$

all inclusions are geometrically clear with the exception of the last one. To ensure that one, we require

$$1 + h\frac{\alpha+1/\alpha}{2} \leq \frac{\varrho-1/\varrho}{2}. \quad (5.7)$$

Inserting the condition  $\alpha = (1 - \varepsilon)\varrho/h$  and rearranging terms, we see that (5.7) is true if we ensure

$$\varepsilon\varrho^2 \geq 2\varrho + 1 + \frac{h^2}{1-\varepsilon}. \quad (5.8)$$

In view of  $h \in [0, 2]$ , this last condition is certainly met if

$$\varrho \geq \varrho_0 := \frac{1 + \sqrt{1 + \varepsilon(1 + 4/(1 - \varepsilon))}}{\varepsilon},$$

which finishes the argument.  $\square$

**Lemma 5.4** *Fix  $0 < \bar{q} < 1$  and  $\gamma > 0$ . Choose  $q \in (\bar{q}, 1)$ . Then there is  $m_0 > 0$  depending only on  $\bar{q}$ ,  $\gamma$ , and the chosen  $q$  such that the following is true:*

*Let  $J_1 \subset J_0$  be two closed intervals with  $|J_1|/|J_0| \leq \bar{q} < 1$ . Denote  $h_0 := |J_0|/2$ ,  $h_1 := |J_1|/2$ . Let  $\kappa, c_0, c_1 \in \mathbb{R}$  and assume that*

$$|\kappa h_0(c_0 - c_1)| \leq \gamma.$$

*Then for all  $m \geq m_0$  and all  $\pi \in \Pi_m$*

$$\inf_{v \in \Pi_m} \|\exp(\mathbf{i}\kappa c_0 \cdot) \pi - \exp(\mathbf{i}\kappa c_1 \cdot) v\|_{\infty, J_1} \leq q^m \|\pi\|_{\infty, J_0}.$$

*Proof.* Let  $\Phi := \Phi_{J_0} : [-1, 1] \rightarrow J_0$  be the orientation preserving affine bijection as in Section 2.1. Set  $\hat{\pi} := \pi \circ \Phi$ ,  $[a, b] := \hat{J}_1 := \Phi^{-1}(J_1)$ . Set  $h := h_1/h_0 = (b-a)/2 \leq \bar{q}$ . We have

$$\inf_{v \in \Pi_m} \|\exp(\mathbf{i}\kappa c_0 \cdot) \pi - \exp(\mathbf{i}\kappa c_1 \cdot) v\|_{\infty, J_1} = \inf_{v \in \Pi_m} \|\exp(\mathbf{i}\kappa h_0(c_0 - c_1) \cdot) \hat{\pi} - v\|_{\infty, \hat{J}_1}.$$

By the polynomial approximation results of Lemma 3.11, we estimate for arbitrary  $\alpha > 1$  and  $m \in \mathbb{N}_0$

$$\begin{aligned} \inf_{v \in \Pi_m} \|\exp(\mathbf{i}\kappa h_0(c_0 - c_1) \cdot) \hat{\pi} - v\|_{\infty, \hat{J}_1} &\leq \frac{2\alpha^{-m}}{\alpha - 1} \|\exp(\mathbf{i}\kappa h_0(c_0 - c_1) \cdot) \hat{\pi}\|_{\infty, \mathcal{D}_\alpha^{a,b}} \\ &\leq \frac{2\alpha^{-m}}{\alpha - 1} \exp\left(|\kappa h_0(c_0 - c_1)| h \frac{\alpha - 1/\alpha}{2}\right) \|\hat{\pi}\|_{\infty, \mathcal{D}_\alpha^{a,b}}. \end{aligned}$$

We now choose  $\alpha$  in dependence on  $m$ . Fix  $\varepsilon \in (0, 1 - \bar{q}/q)$  (so that  $\bar{q}/(1 - \varepsilon) < q$ ) and choose  $\beta > 0$  such that (for the  $q$  of the statement of the lemma)

$$q = \frac{\bar{q}}{1 - \varepsilon} \exp\left(\frac{\gamma(1 - \varepsilon)\beta}{2}\right).$$

We set  $\varrho = \beta m$  and  $\alpha = (1 - \varepsilon)\varrho/\bar{q} = (1 - \varepsilon)\beta m/\bar{q}$ . Lemma 5.3 implies  $\mathcal{D}_\alpha^{a,b} \subset \mathcal{D}_\varrho$  if  $\beta m = \varrho \geq \varrho_0$ . We note that this condition imposes  $m \geq \varrho_0/\beta$ . Furthermore, the Bernstein estimate [13, Thm. 2.2, Chap. 4], gives

$$\|\hat{\pi}\|_{\infty, \mathcal{D}_\alpha^{a,b}} \leq \|\hat{\pi}\|_{\infty, \mathcal{D}_\varrho} \leq \varrho^m \|\hat{\pi}\|_{\infty, (-1,1)}.$$

Hence we obtain

$$\begin{aligned} \inf_{v \in \Pi_m} \|\exp(\mathbf{i}\kappa h_0(c_0 - c_1) \cdot) \hat{\pi} - v\|_{\infty, \hat{J}_1} &\leq \frac{2}{\alpha - 1} \left(\frac{\varrho}{\alpha}\right)^m \exp\left(\gamma h \frac{\alpha - 1/\alpha}{2}\right) \|\hat{\pi}\|_{\infty, (-1,1)} \\ &\leq \frac{2}{\alpha - 1} \left(\frac{\bar{q}}{1 - \varepsilon}\right)^m \exp\left(\frac{\gamma \bar{q} \alpha}{2}\right) \|\hat{\pi}\|_{\infty, (-1,1)} \\ &= \frac{2}{\alpha - 1} \left(\frac{\bar{q}}{1 - \varepsilon}\right)^m \exp\left(m \frac{\gamma(1 - \varepsilon)\beta}{2}\right) \|\hat{\pi}\|_{\infty, (-1,1)} \\ &= \frac{2}{\alpha - 1} \left(\frac{\bar{q}}{1 - \varepsilon} \exp\left(\frac{\gamma(1 - \varepsilon)\beta}{2}\right)\right)^m \|\hat{\pi}\|_{\infty, (-1,1)} = \frac{2}{\alpha - 1} q^m \|\hat{\pi}\|_{\infty, (-1,1)} \\ &\leq q^m \|\hat{\pi}\|_{\infty, (-1,1)}, \end{aligned}$$

where, in the last step we used that  $\alpha \rightarrow \infty$  as  $m \rightarrow \infty$ ; more precisely, we ensure  $\alpha \geq 3$  by requiring

$$m \geq m_0 := \max\left\{\frac{\varrho_0}{\beta}, \left\lceil 3 \frac{\bar{q}}{(1 - \varepsilon)\beta} \right\rceil\right\}.$$

□



**Lemma 5.5 (Interpolation error)** *Let  $\ell \in [1 : L]$  and  $\pi \in \Pi_m$ . We have*

$$\begin{aligned} & \|\exp(\mathbf{i}\kappa c_{\ell-1}\cdot)\pi - \mathfrak{I}_\ell[\exp(\mathbf{i}\kappa c_{\ell-1}\cdot)\pi]\|_{\infty, J_\ell} \\ & \leq (1 + \Lambda_m) \inf_{v \in \Pi_m} \|\exp(\mathbf{i}\kappa c_{\ell-1}\cdot)\pi - \exp(\mathbf{i}\kappa c_\ell\cdot)v\|_{\infty, J_\ell}. \end{aligned} \quad (5.9)$$

*Proof.* Let  $v \in \Pi_m$  be arbitrary. Write

$$\begin{aligned} & \exp(\mathbf{i}\kappa c_{\ell-1}\cdot)\pi - \mathfrak{I}_\ell[\exp(\mathbf{i}\kappa c_{\ell-1}\cdot)\pi] \\ & = \exp(\mathbf{i}\kappa c_{\ell-1}\cdot)\pi - \exp(\mathbf{i}\kappa c_\ell\cdot)v \\ & \quad - \exp(\mathbf{i}\kappa c_\ell\cdot)\mathfrak{I}_{J_\ell}[\exp(-\mathbf{i}\kappa c_\ell\cdot)\{\exp(\mathbf{i}\kappa c_{\ell-1}\cdot)\pi - \exp(\mathbf{i}\kappa c_\ell\cdot)v\}]. \end{aligned}$$

Hence, by the stability of  $\mathfrak{I}$  we get

$$\begin{aligned} & \|\exp(\mathbf{i}\kappa c_{\ell-1}\cdot)\pi - \mathfrak{I}_\ell[\exp(\mathbf{i}\kappa c_{\ell-1}\cdot)\pi]\|_{\infty, J_\ell} \\ & \leq (1 + \Lambda_m) \|\exp(\mathbf{i}\kappa c_{\ell-1}\cdot)\pi - \exp(\mathbf{i}\kappa c_\ell\cdot)v\|_{\infty, J_\ell}, \end{aligned}$$

which concludes the proof.  $\square$

**Theorem 5.6 (Stability of reinterpolation)** *Write  $h_\ell = |J_\ell|/2$  for all  $\ell \in [0 : L]$ . Let  $c_0, \dots, c_L \in \mathbb{R}$  be such that*

$$|\kappa h_{\ell-1}(c_{\ell-1} - c_\ell)| \leq \gamma \quad \text{for all } \ell \in [1 : L]. \quad (5.10)$$

*for some  $\gamma > 0$ . Fix  $q \in (\bar{q}, 1)$ . Then there is  $m_0 > 0$ , depending solely on  $\gamma, \bar{q}$ , and the chosen  $q$ , such that for all  $m \geq m_0$*

$$\|(I - \mathfrak{I}_L \circ \dots \circ \mathfrak{I}_1)[\exp(\mathbf{i}\kappa c_0\cdot)\pi]\|_{\infty, J_L} \leq \varepsilon_{m,L} \|\exp(\mathbf{i}\kappa c_0\cdot)\pi\|_{\infty, J_0} \quad \text{for all } \pi \in \Pi_m, \quad (5.11a)$$

$$\|\mathfrak{I}_\ell\|_{C(J_L) \leftarrow C(J_0)} \leq \Lambda_m (1 + \varepsilon_{m,L}), \quad (5.11b)$$

$$\varepsilon_{m,L} := (1 + (1 + \Lambda_m)q^m)^L - 1. \quad (5.11c)$$

*Let  $\hat{q} \in (q, 1)$ . Then there is  $K > 0$ , depending solely on  $\gamma, \bar{q}$ , the chosen  $\hat{q}$ , and the constants  $C_\Lambda, \lambda$  of (3.3), such that the following implication holds:*

$$m \geq K(1 + \log L) \quad \implies \quad \varepsilon_{m,L} \leq \hat{q}^m. \quad (5.12)$$

*Proof.* Let  $m_0$  be given by Lemma 5.4, and let  $m \geq m_0$ .

*Step 1.* (stability of  $\mathfrak{I}_\ell$ ). Then, combining Lemmas 5.4 and 5.5, the following approximation estimate holds for arbitrary  $\pi \in \Pi_m$  and  $\ell \in [1 : L]$ :

$$\|\exp(\mathbf{i}\kappa c_{\ell-1}\cdot)\pi - \mathfrak{I}_\ell[\exp(\mathbf{i}\kappa c_{\ell-1}\cdot)\pi]\|_{\infty, J_\ell} \leq (1 + \Lambda_m)q^m \|\exp(\mathbf{i}\kappa c_{\ell-1}\cdot)\pi\|_{\infty, J_{\ell-1}}, \quad (5.13a)$$

and the triangle inequality yields the stability estimate

$$\|\mathfrak{I}_\ell[\exp(\mathbf{i}\kappa c_{\ell-1}\cdot)\pi]\|_{\infty, J_\ell} \leq (1 + (1 + \Lambda_m)q^m) \|\exp(\mathbf{i}\kappa c_{\ell-1}\cdot)\pi\|_{\infty, J_{\ell-1}}. \quad (5.13b)$$

*Step 2.* (error estimate) We note the following telescoping sum:

$$\begin{aligned} E_1 &:= I - \mathfrak{J}_L \circ \cdots \circ \mathfrak{J}_1 \\ &= (I - \mathfrak{J}_1) + (I - \mathfrak{J}_2) \circ \mathfrak{J}_1 + (I - \mathfrak{J}_3) \circ \mathfrak{J}_2 \circ \mathfrak{J}_1 + \cdots \\ &\quad + (I - \mathfrak{J}_L) \circ \mathfrak{J}_{L-1} \circ \cdots \circ \mathfrak{J}_1. \end{aligned}$$

Let  $\pi \in \Pi_m$  and note that, for each  $\ell \in [1 : L]$  we can find  $\pi_\ell \in \Pi_m$  with

$$\mathfrak{J}_\ell \circ \mathfrak{J}_{\ell-1} \circ \cdots \circ \mathfrak{J}_1[\exp(\mathbf{i}\kappa c_0 \cdot)\pi] = \exp(\mathbf{i}\kappa c_\ell \cdot)\pi_\ell$$

by definition of  $\mathfrak{J}_\ell$ . Hence, the stability and approximation assertions (5.13a), (5.13b) are applicable, and we arrive at

$$\begin{aligned} \|E_1[\exp(\mathbf{i}\kappa c_0 \cdot)\pi]\|_{\infty, J_L} &\leq \sum_{\ell=1}^L (1 + \Lambda_m)q^m (1 + (1 + \Lambda_m)q^m)^{\ell-1} \|\exp(\mathbf{i}\kappa c_0 \cdot)\pi\|_{\infty, J_0} \\ &= (1 + \Lambda_m)q^m \frac{(1 + (1 + \Lambda_m)q^m)^L - 1}{(1 + \Lambda_m)q^m} \|\exp(\mathbf{i}\kappa c_0 \cdot)\pi\|_{\infty, J_0} \\ &= \{(1 + (1 + \Lambda_m)q^m)^L - 1\} \|\exp(\mathbf{i}\kappa c_0 \cdot)\pi\|_{\infty, J_0}, \end{aligned}$$

which shows (5.11a).

*Step 3.* (stability estimate) Inspection shows that (5.11b) is valid for  $L = 1$ . For  $L \geq 2$ , we consider  $u \in C(J_0)$  and define  $\pi_0 \in \Pi_m$  by  $\pi_0 := \mathfrak{J}_{J_0}[\exp(-\mathbf{i}\kappa c_0 \cdot)u]$ . By definition of  $\mathfrak{J}_0$ , we have

$$\begin{aligned} \mathfrak{J}_0[u] &= \exp(\mathbf{i}\kappa c_0 \cdot)\pi_0, \quad \|\pi_0\|_{\infty, J_0} \leq \Lambda_m \|u\|_{\infty, J_0}, \\ \mathfrak{J}_\mathfrak{E}[u] &= \mathfrak{J}_L \circ \cdots \circ \mathfrak{J}_1[\exp(\mathbf{i}\kappa c_0 \cdot)\pi_0]. \end{aligned}$$

Using (5.11a), we obtain

$$\begin{aligned} \|\mathfrak{J}_\mathfrak{E}[u]\|_{\infty, J_L} &= \|\mathfrak{J}_L \circ \cdots \circ \mathfrak{J}_1[\exp(\mathbf{i}\kappa c_0 \cdot)\pi_0]\|_{\infty, J_L} \\ &= \|\exp(\mathbf{i}\kappa c_0 \cdot)\pi_0 - E_1[\exp(\mathbf{i}\kappa c_0 \cdot)\pi_0]\|_{\infty, J_L} \\ &\leq \|\pi_0\|_{\infty, J_L} + \varepsilon_{m,L} \|\pi_0\|_{\infty, J_0} \leq (1 + \varepsilon_{m,L})\Lambda_m \|u\|_{\infty, J_0}, \end{aligned}$$

and this is (5.11b).

*Step 4.* (bound for  $\varepsilon_{m,L}$ ) Let  $\tilde{q} \in (q, \hat{q})$ . The stability assumption (3.3) implies that we can find  $m_1$  such that

$$(1 + \Lambda_m)q^m \leq (1 + C_\Lambda(m+1)^\lambda)q^m \leq \tilde{q}^m \quad \text{for all } m \geq m_1,$$

so we obtain

$$\varepsilon_{m,L} \leq (1 + \tilde{q}^m)^L - 1 = (1 + \tilde{q}^m)^{\tilde{q}^{-m}\tilde{q}^m L} - 1 \leq \exp(L\tilde{q}^m) - 1,$$

where we used  $\sup_{x>0}(1+x)^{1/x} \leq e$ . Using the estimate  $\exp(x) - 1 \leq ex$ , which is valid for  $x \in [0, 1]$ , and assuming  $\tilde{q}^m L \leq 1$  (note that this holds for  $m \geq K(1 + \log L)$  with  $K \geq 1/|\log \tilde{q}|$ ), we obtain

$$\begin{aligned} \tilde{q}^{-m\varepsilon_{m,L}} &\leq e\tilde{q}^{-m}\tilde{q}^m L = e(\tilde{q}/\hat{q})^m L = \exp(\log L + \log e - m \log(\hat{q}/\tilde{q})) \\ &\leq \exp(\log L + \log e - K \log L \log(\hat{q}/\tilde{q}) - K \log(\hat{q}/\tilde{q})). \end{aligned}$$

Choosing  $K := \max\{m_0, m_1, 1/\log(\hat{q}/\tilde{q}), 1/|\log \tilde{q}|\}$  completes the proof.  $\square$

### 5.3. Multidimensional nested interpolation

Using Theorem 5.6, we can investigate the stability and approximation properties of the multidimensional directional interpolation operator. We recall (5.5), i.e.,

$$\mathfrak{I}_{\tau_L, c_L} \circ \cdots \circ \mathfrak{I}_{\tau_0, c_0} = (\mathfrak{I}_{L,1} \circ \cdots \circ \mathfrak{I}_{0,1}) \otimes \cdots \otimes (\mathfrak{I}_{L,n} \circ \cdots \circ \mathfrak{I}_{0,n}).$$

**Theorem 5.7 (Nested directional interpolation)** *Let  $\bar{q} \in (0, 1)$  and  $\gamma \in \mathbb{R}_{>0}$  such that*

$$\frac{b_{\ell, \iota} - a_{\ell, \iota}}{b_{\ell-1, \iota} - a_{\ell-1, \iota}} \leq \bar{q} \quad \text{for all } \ell \in [1 : L], \iota \in [1 : n] \quad (5.14)$$

and

$$\kappa \operatorname{diam}(\tau_{\ell-1}) \|c_{\ell-1} - c_\ell\| \leq \gamma \quad \text{for all } \ell \in [1 : L] \quad (5.15)$$

hold. Let  $q \in (\bar{q}, 1)$  and  $\hat{q} \in (q, 1)$ . There is  $K$ , depending solely on  $\gamma, \bar{q}$ , the chosen  $q$ , as well as  $C_\Lambda$  and  $\lambda$  of (3.3) such that for all  $m \geq K(1 + \log L)$  we have

$$\|\mathfrak{I}_{\tau_L, c_L} \circ \cdots \circ \mathfrak{I}_{\tau_0, c_0}\|_{\infty, \tau_L \leftarrow \tau_0} \leq \Lambda_m^n (1 + \hat{q}^m)^n.$$

*Proof.* We can apply Theorem 5.6 to get

$$\begin{aligned} \|\mathfrak{I}_{\tau_L, c_L} \circ \cdots \circ \mathfrak{I}_{\tau_0, c_0}\|_{\infty, \tau_L \leftarrow \tau_0} &\leq \prod_{\iota=1}^n \|\mathfrak{I}_{L, \iota} \circ \cdots \circ \mathfrak{I}_{0, \iota}\|_{\infty, [a_{L, \iota}, b_{L, \iota}] \leftarrow [a_{0, \iota}, b_{0, \iota}]} \\ &\leq \Lambda_m^n (1 + \varepsilon_{m, L})^n \leq \Lambda_m^n (1 + \hat{q}^m)^n \end{aligned}$$

for all  $m \geq K(1 + \log L)$ .  $\square$

**Corollary 5.8 (Re-interpolated Lagrange polynomials)** *Let  $\bar{q}, \gamma, q, \hat{q}$ , and  $K$  be as in Theorem 5.7, and assume again (5.14) and (5.15). There is a constant  $C_{\lg}$  depending only on  $n$  such that for all  $m \geq K(1 + \log L)$  we have for the functions  $\tilde{L}_{\tau c, \nu} = \tilde{L}_{\tau_0 c_0, \nu}$  of (2.14)*

$$\|\tilde{L}_{\tau_0 c_0, \nu} - L_{\tau_0 c_0, \nu}\|_{\infty, \tau_L} \leq C_{\lg} \hat{q}^m \|L_{\tau_0 c_0, \nu}\|_{\infty, \tau_0} \quad \text{for all } \nu \in M.$$

*Proof.* We have

$$L_{\tau_0 c_0, \nu} = (\exp(\mathbf{i} \kappa c_{0,1} \cdot) L_{[a_{0,1}, b_{0,1}], \nu_1}) \otimes \cdots \otimes (\exp(\mathbf{i} \kappa c_{0,n} \cdot) L_{[a_{0,n}, b_{0,n}], \nu_n})$$

with the transformed univariate Lagrange polynomials  $L_{[a,b], \mu}$  defined in (2.2). We use a telescoping sum to handle the dimensions  $\iota \in [1 : n]$  and apply (5.11a).  $\square$

**Remark 5.9** *As observed in connection with (3.19), we choose  $c_\ell = 0$  if the boxes are sufficiently small relative to  $\kappa$ . In this case, the functions  $\tilde{L}_{\tau c, \nu}$  are standard polynomials and the re-interpolation does not incur any error. In other words: if (3.19) holds,  $L = \mathcal{O}(\log \kappa)$  so that the condition  $m \geq K(1 + \log L)$  reduces to  $m \geq K' \log(\log \kappa)$  for some  $K'$ .*

$\eta_1 = 10, \eta_2 = 1$									
$N$	$\kappa$	$\ G\ _2$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m = 7$	
$9 \times 2^9$	6	8.2 <sub>-4</sub>	1.4 <sub>-7</sub>	7.3 <sub>-9</sub>	3.2 <sub>-10</sub>	1.4 <sub>-11</sub>	4.4 <sub>-13</sub>	2.0 <sub>-14</sub>	0.05
$2 \times 2^{12}$	8	3.7 <sub>-4</sub>	1.9 <sub>-6</sub>	2.3 <sub>-7</sub>	2.5 <sub>-8</sub>	2.3 <sub>-9</sub>	1.8 <sub>-10</sub>	1.3 <sub>-11</sub>	0.07
$9 \times 2^{11}$	12	1.3 <sub>-4</sub>	3.0 <sub>-6</sub>	8.3 <sub>-7</sub>	1.8 <sub>-7</sub>	3.3 <sub>-8</sub>	5.1 <sub>-9</sub>	7.2 <sub>-10</sub>	0.14
$2 \times 2^{14}$	16	5.8 <sub>-5</sub>	4.2 <sub>-6</sub>	1.4 <sub>-6</sub>	3.9 <sub>-7</sub>	1.1 <sub>-7</sub>	2.5 <sub>-8</sub>	5.2 <sub>-9</sub>	0.21
$9 \times 2^{13}$	24	2.0 <sub>-5</sub>	3.1 <sub>-6</sub>	1.2 <sub>-6</sub>	3.7 <sub>-7</sub>	9.7 <sub>-8</sub>			(0.26)
$2 \times 2^{16}$	32	9.2 <sub>-6</sub>	2.0 <sub>-6</sub>	9.5 <sub>-7</sub>	3.4 <sub>-7</sub>				(0.36)
$\eta_1 = 10, \eta_2 = 2$									
$N$	$\kappa$	$\ G\ _2$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m = 7$	
$9 \times 2^9$	6	8.2 <sub>-4</sub>	4.2 <sub>-6</sub>	5.8 <sub>-7</sub>	5.5 <sub>-8</sub>	5.5 <sub>-9</sub>	4.7 <sub>-10</sub>	3.6 <sub>-11</sub>	0.08
$2 \times 2^{12}$	8	3.7 <sub>-4</sub>	1.8 <sub>-5</sub>	4.5 <sub>-6</sub>	1.3 <sub>-6</sub>	3.3 <sub>-7</sub>	7.4 <sub>-8</sub>	1.5 <sub>-8</sub>	0.20
$9 \times 2^{11}$	12	1.3 <sub>-4</sub>	1.9 <sub>-5</sub>	6.9 <sub>-6</sub>	1.8 <sub>-6</sub>	4.3 <sub>-7</sub>	9.5 <sub>-8</sub>	1.8 <sub>-8</sub>	0.19
$2 \times 2^{14}$	16	5.8 <sub>-5</sub>	1.2 <sub>-5</sub>	5.0 <sub>-6</sub>	1.7 <sub>-6</sub>	5.1 <sub>-7</sub>	1.3 <sub>-7</sub>	3.1 <sub>-8</sub>	0.24
$9 \times 2^{13}$	24	2.0 <sub>-5</sub>	5.3 <sub>-6</sub>	2.8 <sub>-6</sub>	1.4 <sub>-6</sub>	5.5 <sub>-7</sub>	1.9 <sub>-7</sub>		(0.35)
$2 \times 2^{16}$	32	9.2 <sub>-6</sub>	3.3 <sub>-6</sub>	1.8 <sub>-6</sub>	8.4 <sub>-7</sub>				(0.47)

Table 1: Approximation errors  $\|G - \tilde{G}\|_2$  for the unit sphere

## 6. Numerical experiments

In order to investigate how accurately our theoretical results predict the convergence of an actual implementation of our nested interpolation scheme, we have implemented a “pure” version of the algorithm outlined in Section 2.2.2, i.e., a version that does not use adaptive techniques to improve the compression rate. While we acknowledge that for practical applications an algebraic recompression scheme [18, 2, 4, 5] is crucial, we have chosen this approach to avoid pitfalls like unrealistically low errors due to full rank “approximations”.

We satisfy the admissibility condition (3.18b) by assigning each level  $\ell$  of the cluster tree a set  $\mathcal{D}_\ell$  of directions constructed as follows: we denote the maximal diameter of all clusters on level  $\ell$  by  $\delta_\ell$  and split the surface of the cube  $[-1, 1]^3$  into squares with diameter  $\leq 2\eta_1/(\kappa\delta_\ell)$ . The midpoints  $\tilde{c}$  of these squares are then projected by  $c := \tilde{c}/\|\tilde{c}\|$  to the unit sphere. By construction, each point on the cube’s surface has a distance of less than  $\eta_1/(\kappa\delta_\ell)$  to one of the midpoints, and the projection cannot increase this distance.

We use the unit sphere as the surface  $\Gamma$  for our test, approximated by a triangular mesh constructed by regularly subdividing the faces of a double pyramid and scaling the resulting vertices to move them to the unit sphere. We use meshes with  $N \in \{4608, 8192, 18432, 32768, 73728, 131072\}$  triangles.

The cluster tree is set up by geometrical bisection. The algorithm stops subdividing clusters  $\tau$  as soon as the corresponding index set  $\hat{\tau}$  contains not more than 64 indices.

The wave number  $\kappa$  is chosen to provide us with a high-frequency problem: we have

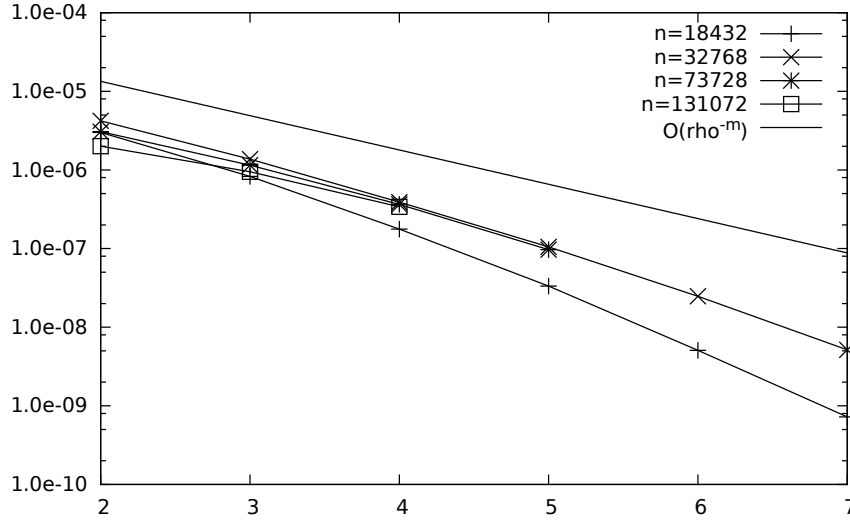


Figure 3: Approximation errors for different meshes and different interpolation degrees

$\kappa h \approx 0.6$ , where  $h$  denotes the maximal mesh width, i.e., we have approximately ten mesh elements per wavelength.

The approximation  $\tilde{G}$  constructed by our algorithm is compared to the original matrix  $G$ , and the spectral norm  $\|G - \tilde{G}\|_2$  of the error is approximated by 20 steps of the power iteration applied to the matrix  $(G - \tilde{G})^*(G - \tilde{G})$ . The norm  $\|G\|_2$  is approximated in the same way.

Table 1 summarizes our results: the rows correspond to the different meshes, while the columns give the spectral error estimates for different interpolation degrees  $m \in [2 : 7]$ . Missing numbers correspond to experiments that did not fit into our machine's memory.

The last column of Table 1 gives the quotient of the last two computed errors, and we expect it to be a good approximation of the asymptotic convergence rate.

We investigate two choices for the admissibility parameters  $\eta_1$  and  $\eta_2$ : for  $\eta_1 = 10$ ,  $\eta_2 = 1$ , our theory predicts an asymptotic convergence rate of  $1/(\sqrt{2}+1) \approx 0.41$ . We can see that the convergence rates in Table 1 appear to converge to this theoretical bound. This is also illustrated in Figure 3 showing the measured errors for the four finest meshes and the slope predicted by our analysis.

For the second choice  $\eta_1 = 10$ ,  $\eta_2 = 2$ , we only expect a convergence rate of  $1/(\sqrt{5/4}+1/2) \approx 0.52$ . Once again, the measured rates are bounded by the predicted rate.

Table 2 lists the results for the surface of the cube  $[-1, 1]^3$  instead of the sphere, discretized with  $N \in \{6912, 12288, 27648, 49152, 110592\}$  triangles. The convergence rates are very similar, illustrating that the directional interpolation does not rely on the smoothness of the surface  $\Gamma$ .

We conclude that our error estimate appears to be quite accurate, particularly for smaller values of  $\eta_2$ , i.e., for more restrictive admissibility conditions.

		$\eta_1 = 10, \eta_2 = 1$								
$N$	$\kappa$	$\ G\ _2$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m = 7$		
$27 \times 2^8$	6	1.0 <sub>-3</sub>	5.8 <sub>-6</sub>	4.3 <sub>-7</sub>	2.4 <sub>-8</sub>	1.4 <sub>-9</sub>	8.2 <sub>-11</sub>	4.6 <sub>-12</sub>	0.06	
$3 \times 2^{12}$	8	4.6 <sub>-4</sub>	5.3 <sub>-6</sub>	5.8 <sub>-7</sub>	5.8 <sub>-8</sub>	4.2 <sub>-9</sub>	3.0 <sub>-10</sub>	1.7 <sub>-11</sub>	0.06	
$27 \times 2^{10}$	12	1.5 <sub>-4</sub>	1.4 <sub>-5</sub>	4.3 <sub>-6</sub>	1.1 <sub>-6</sub>	2.8 <sub>-7</sub>	5.1 <sub>-8</sub>	7.2 <sub>-9</sub>	0.14	
$3 \times 2^{14}$	16	7.1 <sub>-5</sub>	1.0 <sub>-5</sub>	4.5 <sub>-6</sub>	1.3 <sub>-6</sub>	4.1 <sub>-7</sub>	1.1 <sub>-7</sub>	2.6 <sub>-8</sub>	0.24	
$27 \times 2^{12}$	24	2.5 <sub>-5</sub>	5.4 <sub>-6</sub>	2.5 <sub>-6</sub>	1.1 <sub>-6</sub>	4.2 <sub>-7</sub>			(0.38)	
		$\eta_1 = 10, \eta_2 = 2$								
$N$	$\kappa$	$\ G\ _2$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m = 7$		
$27 \times 2^8$	6	1.0 <sub>-3</sub>	2.4 <sub>-5</sub>	7.4 <sub>-6</sub>	2.0 <sub>-6</sub>	4.7 <sub>-7</sub>	8.4 <sub>-8</sub>	1.1 <sub>-8</sub>	0.13	
$3 \times 2^{12}$	8	4.6 <sub>-4</sub>	3.8 <sub>-5</sub>	1.6 <sub>-5</sub>	2.8 <sub>-6</sub>	8.2 <sub>-7</sub>	2.1 <sub>-7</sub>	4.4 <sub>-8</sub>	0.21	
$27 \times 2^{10}$	12	1.5 <sub>-4</sub>	3.5 <sub>-5</sub>	1.3 <sub>-5</sub>	3.8 <sub>-6</sub>	9.6 <sub>-7</sub>	2.0 <sub>-7</sub>	4.1 <sub>-8</sub>	0.21	
$3 \times 2^{14}$	16	7.1 <sub>-5</sub>	3.4 <sub>-5</sub>	1.7 <sub>-5</sub>	6.2 <sub>-6</sub>	2.1 <sub>-6</sub>	6.2 <sub>-7</sub>	1.5 <sub>-7</sub>	0.24	
$27 \times 2^{12}$	24	2.5 <sub>-5</sub>	1.1 <sub>-5</sub>	8.0 <sub>-6</sub>	4.5 <sub>-6</sub>	2.2 <sub>-6</sub>			(0.49)	

Table 2: Approximation errors  $\|G - \tilde{G}\|_2$  for the surface of the cube  $[-1, 1]^3$

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## A. $\mathcal{DH}^2$ -matrices

Our results can be used to prove convergence of various efficient algorithms for the Helmholtz boundary integral equation. We now discuss a straightforward approach that leads to what we call *directional  $\mathcal{H}^2$ -matrices*, or short  $\mathcal{DH}^2$ -matrices. These matrices are a generalization of the  $\mathcal{H}^2$ -matrix representation [16, 7, 5].

Our definition of  $\mathcal{DH}^2$ -matrices is not identical to the one used in [2], since we do not switch to an  $\mathcal{H}$ -matrix representation for the low-frequency case, but use  $\mathcal{H}^2$ -matrix representations for all blocks.

**Definition A.1 (Cluster tree)** *Let  $\mathcal{T}$  be a labeled tree such that the label  $\hat{t}$  of each node  $t \in \mathcal{T}$  is a subset of the index set  $\mathcal{I}$ . We call  $\mathcal{T}$  a cluster tree for  $\mathcal{I}$  if*

- *the root  $r \in \mathcal{T}$  is assigned the label  $\hat{r} = \mathcal{I}$ ,*
- *the index sets of siblings are disjoint, i.e.,*

$$t_1 \neq t_2 \implies \hat{t}_1 \cap \hat{t}_2 = \emptyset \quad \text{for all } t \in \mathcal{T}, \ t_1, t_2 \in \text{sons}(t), \text{ and}$$

- *the index sets of a cluster's sons are a partition of their father's index set, i.e.,*

$$\hat{t} = \bigcup_{t' \in \text{sons}(t)} \hat{t}' \quad \text{for all } t \in \mathcal{T} \text{ with } \text{sons}(t) \neq \emptyset.$$

A cluster tree for  $\mathcal{I}$  is usually denoted by  $\mathcal{T}_{\mathcal{I}}$ . Its nodes are called *clusters*.

A cluster tree  $\mathcal{T}_{\mathcal{I}}$  can be split into levels: we let  $\mathcal{T}_{\mathcal{I}}^{(0)}$  be the set containing only the root of  $\mathcal{T}_{\mathcal{I}}$  and define

$$\mathcal{T}_{\mathcal{I}}^{(\ell)} := \{t' \in \mathcal{T}_{\mathcal{I}} : t' \in \text{sons}(t) \text{ for a } t \in \mathcal{T}_{\mathcal{I}}^{(\ell-1)}\} \quad \text{for all } \ell \in \mathbb{N}.$$

For each cluster  $t \in \mathcal{T}_{\mathcal{I}}$ , there is exactly one  $\ell \in \mathbb{N}_0$  such that  $t \in \mathcal{T}_{\mathcal{I}}^{(\ell)}$ . We call this the *level number* of  $t$  and denote it by  $\text{level}(t) = \ell$ . The maximal level

$$p_{\mathcal{I}} := \max\{\text{level}(t) : t \in \mathcal{T}_{\mathcal{I}}\}$$

is called the *depth* of the cluster tree.

Pairs of clusters  $(t, s)$  correspond to subsets  $\hat{t} \times \hat{s}$  of  $\mathcal{I} \times \mathcal{I}$ , and by extension to submatrices of  $G \in \mathbb{C}^{\mathcal{I} \times \mathcal{I}}$ . These pairs inherit the hierarchical structure provided by the cluster tree.

**Definition A.2 (Block tree)** *Let  $\mathcal{T}$  be a labeled tree, and let  $\mathcal{T}_{\mathcal{I}}$  be a cluster tree for the index set  $\mathcal{I}$  with root  $r_{\mathcal{I}}$ . We call  $\mathcal{T}$  a block tree for  $\mathcal{T}_{\mathcal{I}}$  if*

- *for each  $b \in \mathcal{T}$  there are  $t, s \in \mathcal{T}_{\mathcal{I}}$  such that  $b = (t, s)$ ,*
- *the root  $r \in \mathcal{T}$  satisfies  $r = (r_{\mathcal{I}}, r_{\mathcal{I}})$ ,*



- the label of  $b = (t, s) \in \mathcal{T}$  is given by  $\hat{b} = \hat{t} \times \hat{s}$ , and
- for each  $b = (t, s) \in \mathcal{T}$  we have

$$\text{sons}(b) \neq \emptyset \implies \text{sons}(b) = \text{sons}(t) \times \text{sons}(s).$$

A block tree for  $\mathcal{T}_{\mathcal{I}}$  is usually denoted by  $\mathcal{T}_{\mathcal{I} \times \mathcal{I}}$ . Its nodes are called blocks.

In the following, we assume that a cluster tree  $\mathcal{T}_{\mathcal{I}}$  for the index set  $\mathcal{I}$  and a block tree  $\mathcal{T}_{\mathcal{I} \times \mathcal{I}}$  for  $\mathcal{T}_{\mathcal{I}}$  are given.

We have to identify submatrices, corresponding to blocks, that can be approximated efficiently. Considering the form (2.9) of the matrix entries, we require the approximation  $\tilde{g}_{ts}$  of the kernel function  $g$  to be valid in the entire support of the basis functions  $\varphi_i$  and  $\varphi_j$  for  $i \in \hat{t}$  and  $j \in \hat{s}$ .

**Definition A.3 (Bounding box)** Let  $t \in \mathcal{T}_{\mathcal{I}}$  be a cluster. An axis-parallel box  $\tau \subseteq \mathbb{R}^3$  is called a bounding box for  $t$  if

$$\text{supp}(\varphi_i) \subseteq \tau \quad \text{for all } i \in \hat{t}.$$

In practice we can construct bounding boxes of minimal size by a simple and fast recursive algorithm [6, Example 2.2].

Our approximation scheme (2.7) requires a direction for the plane wave. In order to obtain the optimal order of complexity, we fix a finite set of directions for each level of the cluster tree and introduce a connection between the directions for a cluster  $t$  and the directions for its sons  $t' \in \text{sons}(t)$ .

**Definition A.4 (Hierarchical directions)** A family  $(\mathcal{D}_{\ell})_{\ell=0}^{\infty}$  of finite subsets of  $\mathbb{R}^3$  is called a family of hierarchical directions if

$$\|c\| = 1 \vee c = 0 \quad \text{for all } c \in \mathcal{D}_{\ell}, \ell \in \mathbb{N}_0.$$

A family  $(\text{sd}_{\ell})_{\ell=0}^{\infty}$  of mappings  $\text{sd}_{\ell} : \mathcal{D}_{\ell} \rightarrow \mathcal{D}_{\ell+1}$  is called a family of compatible son mappings if

$$\|c - \text{sd}_{\ell}(c)\| \leq \|c - \tilde{c}\| \quad \text{for all } c \in \mathcal{D}_{\ell}, \tilde{c} \in \mathcal{D}_{\ell+1}, \ell \in \mathbb{N}_0.$$

Given a cluster tree  $\mathcal{T}_{\mathcal{I}}$ , a family of hierarchical directions, and a family of compatible son mappings, we write

$$\mathcal{D}_t := \mathcal{D}_{\text{level}(t)}, \quad \text{sd}_t(c) := \text{sd}_{\text{level}(t)}(c) \quad \text{for all } t \in \mathcal{T}_{\mathcal{I}}, c \in \mathcal{D}_{\text{level}(t)}.$$

**Remark A.5** The “direction”  $c = 0$  is included in Definition A.4 in order to include the low-frequency case in our scheme in a convenient way, cf. (3.19).

**Remark A.6 (Implementation)** In practice, we only have to define  $\mathcal{D}_{\ell}$  for  $\ell \leq p_{\mathcal{I}}$  and  $\text{sd}_{\ell}$  for  $\ell < p_{\mathcal{I}}$ . Our definition admits infinite levels only to avoid special cases.

In the following, we fix a cluster tree  $\mathcal{T}_{\mathcal{I}}$ , a family  $(\mathcal{D}_\ell)_{\ell=0}^\infty$  of hierarchical directions and a family  $(\text{sd}_\ell)_{\ell=0}^\infty$  of compatible son mappings.

Assume that a block  $b = (t, s) \in \mathcal{T}_{\mathcal{I} \times \mathcal{I}}$  and a direction  $c = c_b \in \mathcal{D}_t = \mathcal{D}_s$  is given. According to (2.10), replacing  $g$  in (2.9) with the directional approximation

$$\tilde{g}_{ts}(x, y) = \sum_{\nu \in M} \sum_{\mu \in M} g_c(\xi_{t,\nu}, \xi_{s,\mu}) L_{tc,\nu}(x) \overline{L_{sc,\mu}(y)} \quad \text{for all } x \in \tau, y \in \sigma$$

yields a low-rank factorization of the form

$$G|_{\hat{t} \times \hat{s}} \approx V_{tc} S_b V_{sc}^*. \quad (\text{A.1})$$

According to (2.12), the directional re-interpolation of the Lagrange polynomials described in (2.11) leads to the nested representation

$$V_{tc}|_{\hat{t}' \times k} \approx V_{t'c'} E_{t'c} \quad (\text{A.2})$$

of the matrices  $V_{tc}$ . This approximation brings about a complexity reduction since only the small matrices  $E_{t'c} \in \mathbb{C}^{k \times k}$  need to be stored instead of  $V_{tc} \in \mathbb{C}^{\hat{t} \times k}$ . The notation  $E_{t'c}$  is well-defined since the father  $t \in \mathcal{T}_{\mathcal{I}}$  is uniquely determined by  $t' \in \text{sons}(t)$  due to the tree structure and the direction  $c' = \text{sd}_t(c) \in \mathcal{D}_{t'}$  is uniquely determined by  $c \in \mathcal{D}_t$  due to our Definition A.4.

**Definition A.7 (Directional cluster basis)** *Let  $M$  be a finite index set, and let  $V = (V_{tc})_{t \in \mathcal{T}_{\mathcal{I}}, c \in \mathcal{D}_t}$  be a family of matrices. We call it a directional cluster basis if*

- $V_{tc} \in \mathbb{C}^{\hat{t} \times M}$  for all  $t \in \mathcal{T}_{\mathcal{I}}$  and  $c \in \mathcal{D}_t$ , and
- there is a family  $E = (E_{t'c})_{t \in \mathcal{T}_{\mathcal{I}}, t' \in \text{sons}(t), c \in \mathcal{D}_t}$  such that

$$V_{tc}|_{\hat{t}' \times k} = V_{t'c'} E_{t'c} \quad \text{for all } t \in \mathcal{T}_{\mathcal{I}}, t' \in \text{sons}(t), c \in \mathcal{D}_t, c' = \text{sd}_t(c). \quad (\text{A.3})$$

The elements of the family  $E$  are called transfer matrices for the directional cluster basis  $V$ , and  $k := \#M$  is called its rank.

We can now define the class of matrices that is the subject of this article: we denote the leaves of the block tree  $\mathcal{T}_{\mathcal{I} \times \mathcal{I}}$  by

$$\mathcal{L}_{\mathcal{I} \times \mathcal{I}} := \{b \in \mathcal{T}_{\mathcal{I} \times \mathcal{I}} : \text{sons}(b) = \emptyset\}.$$

The corresponding sets  $\hat{b} \subseteq \mathcal{I} \times \mathcal{I}$  form a disjoint partition of  $\mathcal{I} \times \mathcal{I}$ , so a matrix  $G$  is uniquely determined by the submatrices  $G|_{\hat{b}}$  for  $b \in \mathcal{L}_{\mathcal{I} \times \mathcal{I}}$ . For most of these submatrices, we can find an approximation of the form (A.1). These matrices are called *admissible* and collected in a subset

$$\mathcal{L}_{\mathcal{I} \times \mathcal{I}}^+ := \{b \in \mathcal{L}_{\mathcal{I} \times \mathcal{I}} : b \text{ is admissible}\}.$$

The remaining blocks are called *inadmissible* and collected in the set

$$\mathcal{L}_{\mathcal{I} \times \mathcal{I}}^- := \mathcal{L}_{\mathcal{I} \times \mathcal{I}} \setminus \mathcal{L}_{\mathcal{I} \times \mathcal{I}}^+.$$

How to decide whether a block is admissible or not is the topic of Section 3.3.

**Definition A.8 (Directional  $\mathcal{H}^2$ -matrix)** Let  $V$  and  $W$  be directional cluster bases for  $\mathcal{T}_{\mathcal{I}}$ . Let  $G \in \mathbb{C}^{\mathcal{I} \times \mathcal{I}}$  be a matrix. We call it a directional  $\mathcal{H}^2$ -matrix (or simply a  $\mathcal{DH}^2$ -matrix) if there are families  $S = (S_b)_{b \in \mathcal{L}_{\mathcal{I} \times \mathcal{I}}^+}$  and  $(c_b)_{b \in \mathcal{L}_{\mathcal{I} \times \mathcal{I}}^+}$  such that

- $S_b \in \mathbb{C}^{k \times k}$  and  $c_b \in \mathcal{D}_t = \mathcal{D}_s$  for all  $b = (t, s) \in \mathcal{L}_{\mathcal{I} \times \mathcal{I}}^+$ , and
- $G|_{\hat{t} \times \hat{s}} = V_{tc} S_b W_{sc}^*$  with  $c = c_b$  for all  $b = (t, s) \in \mathcal{L}_{\mathcal{I} \times \mathcal{I}}^+$ .

The elements of the family  $S$  are called coupling matrices, and  $c_b$  is called the block direction for  $b \in \mathcal{T}_{\mathcal{I} \times \mathcal{I}}$ . The cluster bases  $V$  and  $W$  are called the row cluster basis and column cluster basis, respectively.

A  $\mathcal{DH}^2$ -matrix representation of a matrix  $G$  consists of  $V$ ,  $W$ ,  $S$  and the family  $(G|_{\hat{b}})_{b \in \mathcal{L}_{\mathcal{I} \times \mathcal{I}}^-}$  of nearfield matrices corresponding to the inadmissible leaves of  $\mathcal{T}_{\mathcal{I} \times \mathcal{I}}$ .

Let  $G$  be a  $\mathcal{DH}^2$ -matrix for the directional cluster bases  $V$  and  $W$ , and let  $x \in \mathbb{C}^{\mathcal{I}}$ . We denote the corresponding cluster tree by  $\mathcal{T}_{\mathcal{I}}$  and the corresponding block tree by  $\mathcal{T}_{\mathcal{I} \times \mathcal{I}}$  with admissible leaves  $\mathcal{L}_{\mathcal{I} \times \mathcal{I}}^+$ . For an efficient evaluation of the matrix-vector product  $y = Gx$ , we follow the familiar approach of fast multipole and  $\mathcal{H}^2$ -matrix techniques: since the submatrices are factorized into three terms

$$G|_{\hat{t} \times \hat{s}} = V_{tc} S_b W_{sc}^* \quad \text{for all } b = (t, s) \in \mathcal{L}_{\mathcal{I} \times \mathcal{I}}^+,$$

the algorithm is split into three phases: in the first phase, called the *forward transformation*, we multiply by  $W_{sc}^*$  and compute

$$\hat{x}_{sc} = W_{sc}^* x|_{\hat{s}} \quad \text{for all } s \in \mathcal{T}_{\mathcal{I}}, c \in \mathcal{D}_s; \quad (\text{A.4a})$$

in the second phase, the *coupling step*, we multiply these coefficient vectors by the coupling matrices  $S_b$  and obtain

$$\hat{y}_{tc} := \sum_{\substack{b=(t,s) \in \mathcal{L}_{\mathcal{I} \times \mathcal{I}}^+ \\ c=c_b}} S_b \hat{x}_{sc} \quad \text{for all } t \in \mathcal{T}_{\mathcal{I}}, c \in \mathcal{D}_t; \quad (\text{A.4b})$$

and in the final phase, the *backward transformation*, we multiply by  $V_{tc}$  to get the result

$$y_i = \sum_{\substack{t \in \mathcal{T}_{\mathcal{I}}, c \in \mathcal{D}_t \\ i \in \hat{t}}} (V_{tc} \hat{y}_{tc})_i \quad \text{for all } i \in \mathcal{I}. \quad (\text{A.4c})$$

The first and third phase can be handled efficiently by using the transfer matrices  $E_{t'c}$ : let  $s \in \mathcal{T}_{\mathcal{I}}$  with  $\text{sons}(s) \neq \emptyset$ , and let  $c \in \mathcal{D}_s$ . Due to the structure of the cluster tree, the set  $\{\hat{s}' : s' \in \text{sons}(s)\}$  is a disjoint partition of the index set  $\hat{s}$ . Combined with (A.3), this implies

$$W_{sc}^* x|_{\hat{s}} = \sum_{s' \in \text{sons}(s)} (W_{sc}|_{\hat{s}' \times k})^* x|_{\hat{s}'} = \sum_{s' \in \text{sons}(s)} E_{s'c}^* V_{s'c}^* x|_{\hat{s}'} = \sum_{s' \in \text{sons}(s)} E_{s'c}^* \hat{x}_{s'c},$$

```

procedure forward( $s, x, \text{var } \hat{x}$ );
if sons( $s$ ) =  $\emptyset$  then
  for  $c \in \mathcal{D}_s$  do  $\hat{x}_{sc} \leftarrow W_{sc}^* x|_{\hat{s}}$ 
else begin
  for  $s' \in \text{sons}(s)$  do forward( $s', x, \hat{x}$ );
  for  $c \in \mathcal{D}_s$  do begin
     $\hat{x}_{sc} \leftarrow 0$ ;
    for  $s' \in \text{sons}(s)$  do
       $\hat{x}_{sc} \leftarrow \hat{x}_{sc} + E_{s'c}^* \hat{x}_{s'c'}$ 
    end
  end
end

procedure backward( $t, \text{var } \hat{y}, y$ );
if sons( $t$ ) =  $\emptyset$  then
  for  $c \in \mathcal{D}_s$  do  $y|_{\hat{t}} \leftarrow y|_{\hat{t}} + V_{tc} \hat{y}_{tc}$ 
else begin
  for  $c \in \mathcal{D}_t$  do
    for  $t' \in \text{sons}(t)$  do
       $\hat{y}_{t'c'} \leftarrow \hat{y}_{t'c'} + E_{t'c} \hat{y}_{tc}$ ;
    for  $t' \in \text{sons}(t)$  do backward( $t', \hat{y}, y$ )
  end
end

```

Figure 4: Fast forward and backward transformation

and we can prepare *all* coefficient vectors  $\hat{x}_{sc}$  by the simple recursion given on the left of Figure 4. By similar arguments we find that the third phase can also be handled by the recursion given on the right of Figure 4.

The submatrices corresponding to inadmissible leaves  $b = (t, s) \in \mathcal{L}_{\mathcal{I} \times \mathcal{I}}^-$  are stored as standard arrays and can be evaluated accordingly.

We see that the algorithms use each of the matrices of the  $\mathcal{DH}^2$ -matrix representation exactly once, so the bound provided by Remark 2.1 for the storage requirements yields an  $\mathcal{O}(Nk + \kappa^2 k^2 \log N)$  complexity for the matrix-vector multiplication.